Relativistic Fluids of Topological Defects

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Abstract

A number of papers on the topic of string fluids written by Vitaly Vanchurin and myself are reviewed. [11][12][13][14] A network of Nambu-Goto strings is coarse-grained and the equations for a generalized fluid are derived. Besides the symmetric energy-momentum tensor, the fluid also has a conserved antisymmetric tensor F related to the topological flux of strings. This F tensor obeys the homogeneous Maxwell equations, and there is a topological constraint similar to Gauss's law for magnetism. The fluid is isentropic and pressureless and foliated by two-dimensional submanifolds which can be considered to be worldsheets of macroscopic strings. The macroscopic strings are shown to obey the known equations of motion of a wiggly string.

The fluid can be generalized to have pressure and be foliated by arbitrary current carrying strings by introducing a natural variational principle. An action is constructed as a functional of three scalar fields which can be identified as the Lagrangian coordinates of the fluid. This same variational principle for a specific choice of functional is shown to lead to the equations of magnetohydrodynamics, in which the F tensor above is indeed the electromagnetic tensor. Furthermore a minor modification in the fields varied leads to the equations for a model of vortices in a superfluid.

The effect of dissipation can be introduced by allowing the F tensor and energy-momentum tensor to depart from their equilibrium forms. The condition that entropy must increase restricts the form of the non-equilibrium components of these tensors, and leads to the analogue of the Navier-Stokes equations for a string fluid. Besides terms involving viscosity there are additional terms dependent on the curvature of the lines of flux. In the case of magnetohydrodynamics these additional terms are shown to be equivalent to Ohm's law and the thermoelectric Nernst effect. The condition that the non-equilibrium terms vanish is used to derive conditions for hydrostatic equilibrium that may be useful in astrophysical situations.

Contents

Chapter 1

Introduction

Networks of one-dimensional strings appear in a variety of contexts. In particular, networks of quantized vortex lines appear in turbulent quantum fluids, and networks of cosmic strings may have formed in a symmetry breaking phase transition in the early universe. These networks have been extensively studied using numerical models which track the motion of individual strings in the network, as in for instance the vortex-filament model of Schwartz [2] or the Smith-Vilenkin model for cosmic strings [3]. But for many purposes it may be useful to instead consider a 'macroscopic' perspective in which individual strings are coarse-grained in a fluid approximation.

This is the idea which will be explored in this thesis, which is a review of several papers published by Vitaly Vanchurin and myself during my time at the Duluth campus of the University of Minnesota. The body of each chapter is taken from a separate published paper. In this introduction and in the synopsis sections opening each chapter, I hope to give context to show how the ideas developed over time and how the original papers are understood in terms of later ideas.

1.1 Kinetic theory and fluid equations

The idea underlying this thesis began with Vanchurin's work on a kinetic theory for systems of Nambu-Goto strings [9]. This paper treated the system of strings as a gas of disconnected string segments which undergo two types of "collisions." One type of collision occurs between adjacent segments on the same string through the ordinary

Nambu-Goto dynamics. The other type of collision occurs between segments which are not necessarily on the same string through the process of reconnection.

This situation is very similar to the kinetic theory of gases undergoing collisions in which we can derive the Boltzmann transport equation. This transport equation involves the quantity $f(x, v, t)dV$, which gives the number of particles with velocity v in a small volume dV about the spatial point x. The moments of f are proportional to conserved quantities like the particle number and momentum, and the transport equation can be integrated to give conservation laws that ultimately lead to fluid equations (see for instance [1]).

In close analogy, Vanchurin constructed a transport equation for the system of colliding string segments. In this transport equation, the function $f(x, A, B, t)$ gives the energy density of string segments with a given velocity of right-moving waves A and velocity of left-moving waves B. The key result of [9] which was relevant to the fluid approach described below is that when f has no spatial dependence it settles into an equilibrium state in which the statistics of the two velocities A and B are independent. This result was generalized into a "local equilibrium" principle in which even if the system varies in space the statistics of left and right movers are independent.

Just as with the Boltzmann transport equation, the transport equation in [9] may be integrated to give conservation laws, and then the local equilibrium principle may be invoked to lead to fluid equations. My own role in this research began in trying to understand if the conservation laws derived through the kinetic theory approach agreed with conserved quantities carried by the Nambu-Goto strings themselves. It was clear that some of the conserved quantities were ordinary energy and momentum, but now that there were two velocities A and B, there were three additional conserved currents whose physical interpretation was originally obscure.

The interpretation of the new conserved currents is discussed in detail in Chapter 2, which is based on the paper [11]. This chapter bypasses the kinetic approach and coarse-grains the system directly to lead to macroscopic conservation laws. It turns out that the three extra conserved charges correspond to the components of the vector displacement Δx^i between the endpoints of a string. The vector displacement can be written as the integral of the charge densities $x^i_{,\sigma}$ over the string with spatial coordinate

σ,

$$\Delta x^i = \int x^i_{,\sigma} d\sigma.$$

The charge densities $x^i_{,\sigma}$ can then be shown to be locally conserved on the string worldsheet with current density $-x^i_{,\tau}$ where τ is the temporal coordinate on the string

$$\frac{\partial}{\partial \tau}(x^i_{,\sigma}) + \frac{\partial}{\partial \sigma}(-x^i_{,\tau}) = 0.$$

Even though on the worldsheet this local conservation law holds trivially due to commutation of partial derivatives, it is shown in Chapter 2 to lead to the nontrivial macroscopic conservation law of an antisymmetric tensor F

$$\nabla_\mu F^{\mu\nu} = 0.$$

The physical interpretation of F is that if we integrate the dual of F as a differential two-form over a surface, the result is the net flux of strings through the surface. This statement is expressed in coordinate dependent language in Chapter 2. The understanding in terms of differential forms came later and is expressed in the later chapters of this thesis.

We note already in Chapter 2 that the conservation of F is formally equivalent to the homogeneous Maxwell's equations. Indeed in many models, strings carry magnetic flux and so the net topological flux of strings is proportional to the net magnetic flux. In other words, the F tensor, which was derived without any consideration of dynamics, is directly proportional to the macroscopic electromagnetic field for these models.

The fluid equations of Chapter 2 are based on both conservation of the energy-momentum tensor and this F tensor so it is tempting to make the connection to magnetohydrodynamics (MHD) which involves the conservation of the same tensors. A connection between Nambu-Goto strings and MHD had in fact been previously noticed by Olesen [25]. The connection did not become completely clear in our research until we generalized the fluid equations in the manner described in Chapter 4 and 5 [13][14]. But now it is clear that the main difference between a coarse-grained fluid of local strings and ordinary MHD is in the details of the equation of state.

Even though the fluid equations were developed in the context of the kinetic theory of string segments [9], they do not agree with the equations originally derived in that paper. The discrepancy was eventually resolved in a later paper on the kinetic theory [10]. The resolution required taking into account the independent motion of both the left and right movers on each string segment. The energy of the string segments could be considered as flowing in either the direction of the left movers or the right movers. The consistency condition that either choice lead to the same result turned out to be equivalent to conservation of the F tensor above, which we were previously thinking of as a topological requirement that the strings be connected. It was surprising to us that this requirement would also appear in a model of a gas of disconnected string segments.

But even with this successful resolution, the kinetic model is in need of further work. For one thing it predicts a pressureless state at equilibrium, whereas in the numerical simulations of Smith and Vilenkin [3] a gas of small loops is produced. One may argue that the equilibrium state of the Smith-Vilenkin model is artificial since it depends on a minimum loop size which is imposed as a cutoff. However the kinetic model also imposes a cutoff as a finite fixed size of string segments.

Another objection to the kinetic theory is that it is not clear how it can be related to the dissipative fluid equations derived in Chapter 5 [14]. In principle the transport equation should describe the system even out of local equilibrium and thus should describe dissipative effects. However the notion of entropy S_{KT} in the kinetic theory [9] in terms of the distribution f appearing in the transport equation

$$S_{KT} \propto -\int dA\, dB f(A,B) \log f(A,B),$$

does not agree in general with the notion of entropy discussed in Chapter 5. If put in the same language as the kinetic theory, the latter definition of entropy density S in equilibrium can be shown to be proportional to

$$S \propto \rho(\sqrt{1-|\bar{A}|^2} + \sqrt{1-|\bar{B}|^2}),$$

where ρ is the energy density and $\bar{A}, \bar{B}$ are expectation values of the vector quantities A and B, all of which can be expressed as moments of f.

So although the original idea leading to the work in this thesis was in terms of the kinetic theory, the development of the fluid description has diverged significantly. So since it is off of the main line of development I have chosen to not include the paper [10] on the corrected transport equation as a chapter in this thesis.

1.2 Development of the fluid equations

The derivation of the fluid equations themselves in Chapter 2 and 3 is fairly straightforward. The individual Nambu-Goto strings in the network are described by the velocities of right and left moving waves A and B. Both the energy-momentum tensor T and the topological flux tensor F are found for the individual strings in terms of A and B, and then these tensors are coarse-grained (denoted by angled brackets) to find the T and F tensors for the fluid,

$$T^{\mu\nu} = \langle A^{(\mu} B^{\nu)} \rangle$$
$$F^{\mu\nu} = \langle A^{[\mu} B^{\nu]} \rangle.$$

It turns out that both conserved tensors are just the symmetric and antisymmetric parts of the coarse-grained tensor product of A and B. To find a closed set of equations we make the same assumption of "local equilibrium" discussed above in the context of the kinetic theory. The statistics of A and B in each coarse-graining volume are taken to be uncorrelated, so the tensor product factors into the average velocities $\bar{A}$ and $\bar{B}$ and a normalization factor ρ which is just the energy density of the fluid,

$$\langle A^\mu B^\nu \rangle = \rho \bar{A}^\mu \bar{B}^\nu.$$

The fluid equations then follow from the conservation laws for T and F, which are equivalent to vanishing of the covariant derivative in both indices.

$$\nabla_\mu (\rho \bar{A}^\mu \bar{B}^\nu) = \nabla_\nu (\rho \bar{A}^\mu \bar{B}^\nu) = 0.$$

When these equations are expanded in terms of the fluid velocity $\bar{v}$ and the average string direction $\bar{u}$, they take a form similar to the Euler equations for a perfect fluid, together with a topological constraint

$$\nabla \cdot (\rho \bar{\mathbf{u}}) = 0,$$

which is similar to Gauss's law for magnetic fields.

Covariance and connections to previous work

The procedure discussed thus far is detailed in Chapter 2, and is explicitly frame dependent. The energy density ρ transforms as the $0,0$ component of a tensor, and the fluid velocity and string direction vectors $\bar{v}, \bar{u}$ are averaged with respect to this energy density in a preferred frame.

However the conserved tensors T and F do transform as covariant objects, and indeed soon after the publication of [11] it was discovered that the coarse-graining procedure and the fluid equations can be generalized to an arbitrary background metric. The covariant procedure was first published in the context of the improved kinetic theory [10], and it is described in Chapter 3.

In regards to the development of this thesis, what is most important about the covariant form is that it made the connections to earlier works on string fluids more apparent. As early as 1979, Stachel [32] and Letelier [33] published work on phenomenological models of string fluids (or "string dust") similar to the model described here.

To understand the differences with these models it is important to point out another feature of the string fluid which was also discovered soon after the publication of [11]. If we consider the average string direction vector field $\bar{\mathbf{u}}$, the field lines are one-dimensional objects that in some sense appear like 'macroscopic' strings. On the other hand we may consider the fluid velocity vector field $\bar{\mathbf{v}}$, which determines one-dimensional trajectories of the fluid in time. It turns out that due to Frobenius' theorem these two vector fields define two-dimensional integrable submanifolds that foliate spacetime. In other words, the macroscopic strings (field lines) of the $\bar{\mathbf{u}}$ field move in the direction of the velocity $\bar{\mathbf{v}}$ and trace out independent two-dimensional worldsheets.

One may ask whether the macroscopic strings in the string fluid themselves satisfy equations of motion. For the Stachel-Letelier model, these macroscopic strings themselves satisfy the Nambu-Goto equations of motion. Indeed imposing this condition is how the Stachel-Letelier model was derived. But the model described here in Chapter 3 is slightly more general. Due to the possibility of statistical variance in the underlying string network, the macroscopic strings will obey different equations of motion.

These equations of motion were also appear in the literature on cosmic strings and a string satisfying them is known as a 'wiggly string' [15]. The name arises from the idea that it describes a Nambu-Goto string with small scale 'wiggle' degrees of freedom integrated out. The connection between the string fluid submanifolds and wiggly strings is most easily seen by comparing with the formalism used by Brandon Carter in describing current carrying strings [16]. The description of the string fluid in terms of wiggly strings (and also another limiting case known as 'chiral strings') is made in Chapter 3.

1.3 Variational principle and generalized string fluids

The connection to Carter's formalism was useful in that is suggested another direction of generalization. Instead of a fluid with submanifolds behaving as Nambu-Goto strings or wiggly strings, we might a consider a fluid where the submanifolds behave as arbitrary current carrying strings. The wiggly string itself can be thought of as a Nambu-Goto string carrying a conserved current, which may be understood as the entropy current associated with the wiggles [17][14].

The equations of motion for a single current carrying string can be derived from a variational principle in which the internal energy of the string as a function of the conserved current is treated as a Lagrangian [38]. A similar variational principle for the non-current carrying Stachel-Letelier fluid was also described by Kopczyński, which allowed for a non-zero pressure [31]. These principles suggested the proper form for the Lagrangian of the more general string fluid in Chapter 4, which we call a 'perfect string fluid' [13] in analogy to an ordinary isentropic perfect fluid.

The variational principle of Kopczyński was formulated in terms of variations under diffeomorphisms of spacetime. Instead, one of our guiding motivations was to describe the string fluid as field theory with a Langrangian in a form familiar to cosmologists and high-energy physicists. This was indeed possible by describing both the string flux $\tilde{F}$ tensor and the conserved current $\tilde{n}$ carried by the strings in terms of three scalar fields X, Y, Z :

$$\tilde{F} = dX \wedge dY$$
$$\tilde{n} = dX \wedge dY \wedge dZ.$$

The three scalar fields may be understood as Lagrangian coordinates of the fluid ('Lagrangian' here being used in the fluid dynamical sense). The two-dimensional surfaces with constant values of X and Y are just the macroscopic string submanifolds discussed above.

The Lagrangian in the variational principle is just taken to be an arbitrary functional of $\tilde{F}$ and $\tilde{n}$. When we vary the action by the three scalar fields in the ordinary way we recover the equations for a perfect string fluid. Choosing different functionals for the Lagrangian will change the thermodynamics of the fluid. There will be a different equation of state connecting the internal energy, the pressure, and the tension of the macroscopic strings.

An advantage of this variational principle discussed in Chapter 4 is that it makes it very easy to generalize the fluid in many different directions. For instance, in the definition of $\tilde{F}$ and $\tilde{n}$, the field Z appears distinct from X and Y. If we also treat the fields X and Y assymetrically, we can describe a fluid with macroscopic 'domain wall' submanifolds in addition to strings. For instance consider a Lagrangian which is *only* a function of X such as

$$\mathcal{L} = \partial^\mu X \partial_\mu X.$$

On the one hand this is just the equation for a massless scalar field. But as we show in Chapter 4 it may also be treated as a fluid of individual domain walls interacting under pressure, each distinct domain wall specified by a different value of X.

Finally we note that the variational principle of Chapter 4 serves as a bridge between the model of Chapters 2 and 3 which were formulated in the context of cosmic strings and other very different physical systems. We have already noted above that magnetohydrodynamics may be described as a perfect string fluid for a particular choice of the Lagrangian using the same three scalar fields. Furthermore a slight modification of the variational principle can be used to describe a model of a network of superfluid vortices which was first described by Carter and Langlois [39].

1.4 Dissipative string fluids

The perfect string fluid equations discussed so far are not quite analogous to the Navier-Stokes equations in that entropy is conserved and there are no effects due to viscosity. It was noticed by Vanchurin already in [11] that the string fluid equations are similar to the inviscid Burgers' equation and may similarly break down when characteristic curves intersect leading to shock waves. The inclusion of viscosity terms could regularize this behavior. And physically of course, networks of cosmic strings should show strong irreversible dissipative behavior as small loops and wiggles are formed through intersections of larger cosmic string loops.

A hint towards resolving this was found in the conserved current carried by wiggly strings in Chapter 3. If this current is identified as the entropy current associated with small scale structure, the condition that the entropy must always increase restricts the form of possible phenomenological viscosity terms added to the equations of motion. This method is based on the work of Eckart [18] and Landau and Lifsahitz [20] for ordinary dissipative fluids in general relativity.

The argument leading to the equations for dissipative string fluids is discussed in Chapter 5 [14]. We begin with an arbitrary equilibrium equation of state for the string fluid, which is essentially the Lagrangian discussed in Chapter 4. Even in the dissipative case, the energy momentum tensor T and the flux tensor F are conserved —although they do not take the same form as in equilibrium. The tensors are separated into an equilibrium part and a non-equilibrium part which is further decomposed into components transverse and longitudinal to the timelike fluid velocity u and spacelike string direction w.

In equilibrium, the conservation of T in the u direction

$$u_\mu \nabla_\nu T^{\mu\nu} = 0,$$

can be used to derive the conservation of entropy. In the presence of dissipation this equation still holds, but instead it leads to an equation for the change in entropy in terms of the non-equilibrium parts of the T and F tensors. Requiring that this change be positive enforces that the non-equilibrium parts have a specific form in terms of derivatives of u and w and some additional phenomenological functions which can be interpreted as viscosity coefficients.

And indeed the usual shear and bulk viscosity terms from the Navier-Stokes equations appear in the string fluid equations, although there is the possibility of anisotropic viscosities in the directions transverse and longitudinal to the strings. More interesting is that there are additional entropy producing terms arising from the non-equilibrium parts of the F tensor. These lead to a production of entropy due to the curvature of the macroscopic strings. These terms are discussed in Chapter 5.3.3 in terms of plausible loop producing effects in an underlying cosmic string network.

These new dissipative terms can also be understood from the perspective of MHD, where the F tensor is just the ordinary electromagnetic tensor. The dissipative terms just lead to Ohm's law appearing in the equations of motion, where the electric field picks up a term proportional to the current (via Ampere's law).

$$\mathbf{E}_{\mathrm{Ohm}} \propto \nabla \times \mathbf{B}$$

There is also an additional unexpected term leading to an electric field perpendicular to temperature gradients

$$\mathbf{E}_{\mathrm{Nernst}} \propto \mathbf{B} \times \nabla T.$$

This term does not often appear in ordinary resistive MHD (e.g. in [19]), but it is known as the Nernst effect. If the dissipative string fluid analysis is indeed valid for MHD, this effect is required to be present for entropic reasons.

An additional effect present in the dissipative string fluid is the Fourier law of heat conduction.

$$\dot{T} \propto \partial_w^2 T,$$

where the derivative is taken in the w direction parallel to the strings. While this is a physical effect we would expect in a dissipative theory, it is problematic for a causal theory in that it is a parabolic equation. A small change in the temperature will instantly have some effect arbitrarily far along the string.

However this problem also appears in ordinary dissipative fluids and has been treated by Israel and Stewart [8]. The idea is to allow for higher order phenomenological terms in the definition of the entropy current. These additional terms may modify the Fourier law to a hyperbolic equation where temperature changes propagate causally as a wave. In Chapter 5.3.5 it is shown that if there is no additional conserved current in the string fluid, the speed of this temperature wave is equal to the longitudinal speed of sound on a non-dissipative string.

Finally it is shown in Chapter 5.3.4 that the dissipative string fluid equations can be applied to a system that has settled to equilibrium. The condition that no further entropy be produced leads to stronger conditions than a perfect string fluid. For instance there must exist a timelike Killing vector which defines a direction in which the gravitational field does not change, a condition which is also true in hydrostatic equilibrium in an ordinary dissipative fluid [21]. What is new for the string fluid is that there is also a condition which relates the curvature of strings or lines of flux to gradients in temperature and chemical potential. It is hoped that these conditions may be usefully applied to astrophysical systems in equilibrium which may contain a conserved flux such as a magnetic field.

Chapter 2

Fluid of Nambu-Goto strings I

2.1 Synopsis

This chapter is taken from the body of the paper 'Fluid Mechanics of Strings.' [11] First, the relevant physics of single Nambu-Goto strings is reviewed in Sec.2.2. Then the conserved currents associated with a string are discussed in Sec.2.3. This section demonstrates the key point of this chapter and the paper it is based on. Besides the conserved energy-momentum tensor, the strings carry a conserved antisymmetric tensor F associated with the topological flux of strings.

Then in Sec.2.4, the system of Nambu-Goto strings is coarse-grained and the conservation of the tensors introduced Sec.2.3 leads to the fluid equations. An additional principle used to simplify these tensors is the requirement that the fluid be in *local equilibrium*. This condition first appears in the context of the transport equation for a gas of string segments [9], and was later understood in Chapter 5 as the requirement that the fluid be isentropic.[14]

Given both the conservation laws and the condition of local equilibrium, a set of fluid equations is derived for both Minkowski space and for the conformally flat FLRW spacetime which is relevant in cosmology. Soon after the publication of the paper it was realized that the fluid equations could in fact be derived in arbitrary background metric. This point along with many other discoveries about these same fluid equations are discussed in the following Chapter 3, which may be read independently. Still, the current chapter and the paper it is based on remain a gentler introduction to the fluid equations.

2.2 Preliminaries

The dynamics of a single string is well-described by the Nambu-Goto action, which can be expressed in terms of generalized worldsheet coordinates ζ^a,

$$S = -\int \sqrt{-h}\, d^2\zeta, \tag{2.1}$$

where for simplicity the string tension is set equal to one. Here, h is the determinant of the metric on the world sheet, which is induced from the metric $g_{\mu\nu}$ by pulling back the mapping into spacetime $x^\mu(\zeta^a)$:

$$h_{ab} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \zeta^a}\frac{\partial x^\nu}{\partial \zeta^b}. \tag{2.2}$$

By varying the action (4.40) with respect to $g_{\mu\nu}$ we find the energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu}\sqrt{-g} = \int d^2\zeta\sqrt{-h}h^{ab}\frac{\partial x^\mu}{\partial \zeta^a}\frac{\partial x^\nu}{\partial \zeta^b}\delta^{(4)}\left(y^\sigma - x^\sigma\right), \tag{2.3}$$

where y^σ is the argument of $T^{\mu\nu}$, and x^σ is again the mapping from the worldsheet into spacetime. (See [40] for details.)

This expression (2.3) can be simplified by fixing our choice of ζ^a. The timelike coordinate will be denoted by τ and the spacelike coordinate by σ. We fix τ to be equal to the spacetime coordinate x^0:

$$x^0(\tau,\sigma) = \tau. \tag{2.4}$$

Then the integration over τ eliminates the temporal part of the delta function in the expression (2.3) for $T^{\mu\nu}$:

$$T^{\mu\nu}\sqrt{-g} = \int d\sigma \tilde{T}^{\mu\nu}(\sigma)\delta^{(3)}\left(y^i - x^i\right). \tag{2.5}$$

The tilde notation $\tilde{T}^{\mu\nu}$ indicates the non-singular part of the integrand. While $T^{\mu\nu}$ is a singular density over spacetime, $\tilde{T}^{\mu\nu}$ is a density over the worldsheet. This notation will be used for other tensor densities of the form (2.5) as well.

Denoting derivatives with respect to τ and σ by dots and primes respectively, we adopt a further gauge condition on the worldsheet coordinates:

$$\dot{\mathbf{x}} \cdot \mathbf{x}' = 0. \tag{2.6}$$

Restricting our consideration to the Friedmann universe in conformal coordinates with metric

$$g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}, \tag{2.7}$$

the energy density is given by

$$\epsilon \equiv \tilde{T}^{00} = \sqrt{\frac{\mathbf{x}'^2}{1-\dot{\mathbf{x}}^2}}. \tag{2.8}$$

If we also define the string velocity $\mathbf{v} \equiv \dot{\mathbf{x}}$, the tangent vector $\mathbf{u} \equiv \epsilon^{-1}\mathbf{x}'$ and the Hubble parameter $\mathcal{H} \equiv \dot{a}/a$, then the equation of motion is found to be

$$\dot{\mathbf{v}} + 2\mathcal{H}(1-\mathbf{v}^2)\mathbf{v} = \epsilon^{-1}\mathbf{u}'. \tag{2.9}$$

The quantity $\mathbf{v}^2 + \mathbf{u}^2$ is a constant of motion which can be fixed by imposing a final gauge condition,

$$\mathbf{v}^2 + \mathbf{u}^2 = 1. \tag{2.10}$$

By applying these gauge conditions (2.4), (2.6) and (2.10) to equation (2.3) we can solve for the non-singular part of the energy-momentum tensor,

$$\tilde{T}^{\mu\nu} = \epsilon\,(v^\mu v^\nu - u^\mu u^\nu). \tag{2.11}$$

Here u and v have timelike components $v^0 = 1$ and $u^0 = 0$. So the energy density $\tilde{T}^{00} = \epsilon$, the momentum density $\tilde{T}^{i0} = \epsilon v^i$, and the spacelike components $\tilde{T}^{ij}$ appear as the momentum current density in the continuity equation for momentum.

2.3 Conserved Currents

2.3.1 Minkowski Space

In order to simplify the analysis of the energy-momentum tensor, we will first restrict our attention to Minkowski spacetime where $\mathcal{H} = 0$ and $\epsilon = 1$. In this special case, the equations of motion (2.9) simplify to the wave equation,

$$\dot{\mathbf{v}} = \mathbf{u}'. \tag{2.12}$$

Moreover, the conservation of the energy momentum tensor in flat spacetime can be expressed using an ordinary divergence, without additional gravitational correction terms,

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.13}$$

However, because of the delta functions in (2.5), it is not immediately clear how to interpret the continuity (or conservation) equation (2.13). We will approach the problem by considering instead the integral form of the differential equation (2.13) over an appropriate choice of enclosing volume. For a general current density j^μ the integral equation is given by,

$$\partial_0 \int j^0 dV = -\int \mathbf{j} \cdot d\mathbf{A}. \tag{2.14}$$

Particles

When $\mathbf{j}$ is in the direction of the velocity $\mathbf{v}$, the situation is much the same as that of a localized particle. So we begin by considering the current density for a single particle,

$$j^\mu = J^\mu \delta^{(3)}(y^i - x^i) \tag{2.15}$$

We choose a volume in (2.14) which contains the particle for some time $\tau < \tau_0$. The particle leaves the volume at time τ_0 and the boundary surface is chosen such that $\mathbf{v}$ is normal at the point where the particle exits.

By integrating (2.14) over a small interval of time $\Delta\tau$, the left hand side becomes the net change in enclosed charge, $-J^0$. We choose our coordinate system with $x_\perp$ in the direction of $\mathbf{v}$, normal to surface. The current $\mathbf{J}$ is also in this normal direction, but in preparation for the more general case we will write $\mathbf{J} \cdot d\mathbf{A} = J_v \, dA$. The integration over area in the flux integral cancels with the other two dimensions in the delta function and (2.14) reduces to

$$\begin{aligned} -J^0 &= -\int \delta^{(3)}(y^i - x^i(\tau)) \, \mathbf{J} \cdot d\mathbf{A} \, d\tau \\ &= -\int J_v \, \delta^{(1)}(y_\perp - x_\perp(\tau)) \, d\tau \\ &= -\int J_v \, \delta^{(1)}(y_\perp - x_\perp) (\frac{dx_\perp}{d\tau})^{-1} dx_\perp \\ &= -J_v \, v^{-1}. \end{aligned} \tag{2.16}$$

Here the factor of $v = dx_\perp/d\tau$ came about by changing our remaining integration variable to $dx_\perp$. So the continuity equation for a localized particle just implies the familiar fact that the non-singular part of the current-density is the charge times the velocity,

$$J_v = J^0 v. \tag{2.17}$$

Strings

In the case of a string, in addition to the current in the direction of v^i it is physically relevant to have a current propagating along the string in the direction of u^i. Even when a piece of string is contained in a volume it may pierce the surface at two or more points, and the flux of the current density at these points contributes extra terms in (2.14).

Nevertheless, the argument for a localized particle can be extended straightforwardly to an infinitessimal piece of string with $J^\mu = \tilde{J}^\mu d\sigma$. In the limit of $\Delta\tau \to 0$, only the terms due to the string discontinuously leaving the volume remain in the continuity equation. So the current in the direction of v^i follows the same expression as before,

$$\tilde{J}^i_v = \tilde{J}^0 v^i. \tag{2.18}$$

To consider the current in the direction of u^i, we will first write an expression for the flux at a point where the string pierces the surface using the general form of the singular current. The coordinate $x_\perp$ again points in the normal direction, and the $\perp$ subscript denotes the $x_\perp$-component of a vector. Then,

$$\begin{aligned}
\int \mathbf{j}\cdot d\mathbf{A} &= \int \delta^{(3)}(y^i - x^i(\sigma))\,\tilde{\mathbf{J}}\cdot d\mathbf{A}\,d\sigma \\
&= \int \delta^{(3)}(y^i - x^i(\sigma))\,\tilde{J}_\perp dA\,d\sigma \\
&= \int \delta^{(1)}(y_\perp - x_\perp(\sigma))\,\tilde{J}_\perp\,d\sigma \\
&= \pm\tilde{J}_\perp\,(\frac{dx_\perp}{d\sigma})^{-1} = \tilde{J}_\perp\,|x'_\perp|^{-1}
\end{aligned} \tag{2.19}$$

Note that the change in variables leads to a negative sign if $dx_\perp/d\sigma$ is negative, hence the use of the absolute value $|x'_\perp|$.

The expression (2.19) can be applied to the continuity equation for the momentum density T^{i0} in (4.22) with $\epsilon = 1$, where the associated current density has a term in the direction of u^k, $\tilde{J}^k_u = -u^i u^k$. We choose a boundary surface surrounding a segment of string such that u^i is normal to the surface at the two points where the string enters and leaves the enclosed volume. The values of sigma at these points are denoted by σ_i and σ_f, respectively. The left-hand side of the continuity equation (2.14) becomes simply,

$$\partial_0 \int T^{i0} dV = \partial_0 \int v^i \, \delta^{(3)}(y^i - x^i)\, d\sigma \, dV = \int \frac{\partial v^i}{\partial \tau}\, d\sigma. \tag{2.20}$$

Since in flat spacetime $\mathbf{u} = \mathbf{x}'$, the component of $\tilde{J}^k_u$ in the normal direction is just $\tilde{J}_\perp = \mp u^i \, |x'|$. So using (2.19), the continuity equation becomes,

$$\begin{aligned} \int \frac{\partial v^i}{\partial \tau}\, d\sigma &= -(\, \tilde{J}_\perp |x'|^{-1} \Big|_{\sigma_f} + \tilde{J}_\perp |x'|^{-1} \Big|_{\sigma_i}) \\ &= u^i(\sigma_f) - u^i(\sigma_i) \end{aligned} \tag{2.21}$$

which is just the equation of motion (2.12) integrated over $d\sigma$. Note that the equation of motion (2.12) has the form of a one-dimensional continuity equation on the worldsheet,

$$\frac{\partial \tilde{J}^0}{\partial \tau} = -\frac{\partial \tilde{J}_\sigma}{\partial \sigma}. \tag{2.22}$$

In this case the charge density $\tilde{J}^0 = v^i$ and the one-dimensional current density $\tilde{J}_\sigma = -u^i$.

In general, given any continuity equation of the form (2.22) we can reverse the previous argument to find the singular current density in spacetime, $\tilde{J}^i_u = \tilde{J}_\sigma x'^i$. This can be combined with (2.18) for $\tilde{J}^i_v$, to find the total current density.

$$\tilde{J}^k = \tilde{J}^0 \dot{x}^k + \tilde{J}_\sigma x'^k \tag{2.23}$$

In particular, the commutation of partial derivatives is a continuity equation of the form (2.22)

$$\frac{\partial}{\partial \tau}\left(\frac{\partial x^i}{\partial \sigma}\right) = -\frac{\partial}{\partial \sigma}\left(-\frac{\partial x^i}{\partial \tau}\right) \tag{2.24}$$

and by (2.23), this implies a conserved singular charge density $\tilde{J}^0 = x'^i$ with an associated current density which we denote,

$$\tilde{F}^{ik} \equiv x'^i \dot{x}^k - \dot{x}^i x'^k. \tag{2.25}$$

The conservation of this charge density depends only on the commutation of the partial derivatives of $x(\tau, \sigma)$ and not on the Nambu-Goto dynamics.

Intersections

For each of the three components x'^i, there is a continuity equation involving the flux of $\tilde{F}^{ik}$. We may consider extending the expression (2.25) to the timelike components,

$$\tilde{F}^{0k} \equiv -x'^k, \tag{2.26}$$

which motivates us to consider the fluxes of x'^k as well.

From the general expression for the flux of a singular current density (2.19), the flux of x'^k at a single intersection point equals,

$$\tilde{J}_\perp \, |x'_\perp|^{-1} = x'_\perp \, |x'_\perp|^{-1} = \pm 1 \tag{2.27}$$

The sign depends on whether $x'_\perp$ is parallel or antiparallel to the normal direction. Considering $\mathbf{x}'$ to specify a direction of motion along the string, the sign depends on whether the string is leaving or entering the volume.

In general, a string may intersect a closed surface at many points. As long as the string does not terminate in the interior (on a topological monopole for topological strings or on a D-brane for fundamental strings), for each point where the string enters the volume there must be another point at which the string leaves. So this means that the sum of the flux over all of these intersection points equals zero. Using (5.14) we can express this as a flux integral of F^{0k} in space,

$$\oint F^{0k} dA_k = 0. \tag{2.28}$$

And so the top row of $F^{\mu\nu}$ also obeys the continuity equation (2.14), with $j^0 = F^{00} = 0$.

Just as a string can not terminate on a monopole in the interior, an intersection point on the surface can not suddenly disappear. An intersection point where a string leaves the volume can only vanish if it converges with a point where the string enters the volume. This suggests a picture in which the intersection points are two-dimensional particles with a charge of either ± 1. A particle can only be created or annihlated in conjunction with an antiparticle of opposite charge. We wish to find the continuity equation for this flux charge.

From our discussion on localized particles, it is clear that the corresponding current is just the charge multiplied by the two-dimensional velocity w^i on the surface. To find this velocity, we choose our coordinate system so that the surface near an intersection point is given by $x^3 = 0$. Similarly to (2.4) which fixes our worldsheet coordinate τ, we introduce a new spatial worldsheet coordinate ζ which is equal to x^3 in the vicinity of the intersection point. Formally, $x^3(\tau', \zeta) = \zeta$ near the intersection point. For clarity, the transformed timelike coordinate is written as τ', even though $\tau' = \tau$. Then,

$$\begin{aligned} w^k \equiv \frac{\partial x^k}{\partial \tau'} &= \frac{\partial \tau}{\partial \tau'}\frac{\partial x^k}{\partial \tau} + \frac{\partial \sigma}{\partial \tau'}\frac{\partial x^k}{\partial \sigma} \\ &= \dot{x}^k + \frac{\partial \sigma}{\partial \tau'} x'^k. \end{aligned} \tag{2.29}$$

Since the partial derivative with respect to τ' is taken at fixed ζ, $w^3 = 0$. Thus,

$$\frac{\partial \sigma}{\partial \tau'} = \frac{\dot{x}^3}{x'^3} \tag{2.30}$$

and by substituting (2.30) into (2.29) we get

$$w^k = \dot{x}^k - \frac{\dot{x}^3}{x'^3} x'^k. \tag{2.31}$$

To find the two-dimensional singular current density we multiply this velocity by a two-dimensional delta function and the appropriate sign of charge. But since as before $\pm 1 = \int x'^3\, \delta(\zeta)\, d\sigma$, this can be 'upgraded' to a three-dimensional string current density by multiplying w^k in (2.31) by x'^3. So the charge density $\tilde{J}^0 = x'^3$ is conserved with current density

$$\tilde{J}^k = w^i x'^3 = x'^3 \dot{x}^k - \dot{x}^3 x'^k. \tag{2.32}$$

But this is just the expression for the current density $\tilde{F}^{ik}$ in (2.25), only now the continuity equation involves flux through a surface rather than integration over a volume.

Abstracting back to the differential form of the continuity equation (2.13), it is easier to see how these two distinct integral continuity equations involving F^{ik} are related. Treating $F^{i\nu}$ as a vector with index i, we can consider the flux through a surface. Again choosing the coordinate system locally so that the normal is in the x^3 direction, $\partial_0 F^{30} + \partial_k F^{3k} = 0$. But the current F^{3k} is clearly perpendicular to the normal $k = 3$ direction, so the current everywhere lies in the tangent space of the surface. So we can use a two-dimensional divergence theorem to bring (2.13) into the form describing the conservation of intersection points discussed above.

2.3.2 Friedmann Space

So far we have been considering how densities on the worldsheet such as $\tilde{T}^{\mu\nu}$ are related to singular densities in spacetime of the form

$$\mathfrak{T}^{\mu\nu} \equiv \int d\sigma \tilde{T}^{\mu\nu} \delta^{(3)} \left(y^i - x^i\right). \tag{2.33}$$

According to (2.5) the stress-energy tensor is related to $\mathfrak{T}^{\mu\nu}$ through a factor of $\sqrt{-g}$. In Friedmann space (2.7) this factor $\sqrt{-g} = a^4$, and so

$$T^{\mu\nu} = a^{-4}\, \mathfrak{T}^{\mu\nu}. \tag{2.34}$$

In general relativity the continuity equation for the energy-momentum tensor involves the covariant divergence,

$$0 = \nabla_\nu T^{\mu\nu} = \partial_\nu T^{\mu\nu} + \Gamma^{\mu}_{\lambda\nu} T^{\lambda\nu} + \Gamma^{\nu}_{\lambda\nu} T^{\mu\lambda}. \tag{2.35}$$

In Friedmann space the connection coefficients $\Gamma^{\mu}_{\lambda\nu}$ all vanish except for

$$\Gamma^{0}_{\mu\mu} = \Gamma^{\mu}_{0\mu} = \Gamma^{\mu}_{\mu 0} = \mathcal{H}. \tag{2.36}$$

So for any value of ν, $\Gamma^{\nu}_{\lambda\nu}$ is nonzero only if $\lambda = 0$. Thus the last term in (2.35) reduces to

$$\Gamma^{\nu}_{\lambda\nu} T^{\mu\lambda} = 4\mathcal{H} T^{\mu 0} = 4\mathcal{H} a^{-4}\, \mathfrak{T}^{\mu 0}. \tag{2.37}$$

Then by differentiating the first term in (2.35), we find

$$\partial_\nu T^{\mu\nu} = a^{-4}\, \partial_\nu \mathfrak{T}^{\mu\nu} - 4\mathcal{H} a^{-4}\, \mathfrak{T}^{\mu 0} \tag{2.38}$$

and the continuity equation (2.35) reduces to

$$\partial_\nu \mathfrak{T}^{\mu\nu} + \Gamma^{\mu}_{\lambda\nu} \mathfrak{T}^{\lambda\nu} = 0. \tag{2.39}$$

Consider the momentum continuity equations, setting $\mu = i$ and using (2.36):

$$\begin{aligned} 0 &= \partial_\nu \mathfrak{T}^{i\nu} + \Gamma^{i}_{0i} \mathfrak{T}^{0i} + \Gamma^{i}_{i0} \mathfrak{T}^{i0} \\ &= \partial_\nu \mathfrak{T}^{i\nu} + 2\mathcal{H} \mathfrak{T}^{i0} \end{aligned} \tag{2.40}$$

As before, this involves the time derivative of a charge density $\tilde{J}^0 = \tilde{T}^{i0}$, and the divergence of a current density $\tilde{J}^k = \tilde{T}^{ik}$. By (4.22),

$$\begin{aligned} \tilde{T}^{ik} &= \epsilon\,(v^i v^k - u^i u^k) \\ &= (\epsilon\, v^i)\dot{x}^k + (-u^i)x'^k, \end{aligned} \tag{2.41}$$

so $\tilde{J}^k$ takes the form of (2.23), leading to a continuity equation on the string. Here the only difference from (2.22) is the gravitational correction term $2\mathcal{H}\tilde{T}^{i0}$ from (2.40):

$$\begin{aligned} 0 &= \frac{\partial \tilde{J}^0}{\partial \tau} + \frac{\partial \tilde{J}_\sigma}{\partial \sigma} + 2\mathcal{H}\tilde{T}^{i0} \\ &= \frac{\partial(\epsilon\, v^i)}{\partial \tau} + \frac{\partial(-u^i)}{\partial \sigma} + 2\mathcal{H}\epsilon v^i \\ &= \epsilon\,\dot{v}^i + (\dot{\epsilon} + 2\mathcal{H}\epsilon)v^i - u'^i \end{aligned} \tag{2.42}$$

So using the relation $\dot{\epsilon} = -2\mathcal{H}\mathbf{v}^2\epsilon$ (see for instance [40]), we recover the equation of motion (2.9) from a different perspective.

Unlike $\tilde{T}^{\mu\nu}$, the conservation of $\tilde{F}^{\mu\nu}$ depends only on topological properties (e.g. (2.24)). So a conservation law of the form (2.14) remains valid in Friedmann space without any gravitational correction terms. Still, $\tilde{F}^{\mu\nu}$ in (2.25) can be written in a form more appropriate to Friedmann space:

$$\begin{aligned} \tilde{F}^{\mu\nu} &= x'^\mu \dot{x}^\nu - \dot{x}^\mu x'^\nu \\ &= \epsilon\,(u^\mu v^\nu - v^\mu u^\nu). \end{aligned} \tag{2.43}$$

2.4 String Fluid

2.4.1 Continuum Description

As we have seen, the singular charge and current densities associated with a small segment $\Delta\sigma$ of string with a given u and v take the form,

$$q(x,u,v) = \tilde{Q}(u,v)\delta^{(3)}(x-y)\Delta\sigma \tag{2.44}$$

where y is the position of the segment and x is the argument of the density function. We now consider a volume ΔV containing many string segments as in Ref. [9]. The number of enclosed segments with parameters u and v is written as $n(x,u,v)\Delta V$. Consider the integral of the charge density q over the coarse-graining volume ΔV. The delta function factor in q serves to count the number of enclosed segments and the integral becomes,

$$\int \tilde{Q}(u,v)\Delta\sigma\, n(x,u,v)\,\Delta V\,du\,dv \tag{2.45}$$

Here $\epsilon\Delta\sigma$ serves to convert the number density to an energy density, which is notated by $f(x,u,v)\equiv\epsilon\Delta\sigma\, n(x,u,v)$. Dividing by the volume ΔV we find the coarse-grained charge density,

$$\langle\tilde{Q}\rangle\equiv\int\epsilon^{-1}\tilde{Q}f(x,u,v)du\,dv. \tag{2.46}$$

Now consider the continuity equation (2.14) involving the current density associated with $\tilde{J}^{\mu}$. When the volume involved is much larger than ΔV, the average values $\langle J^{\mu}\rangle$ may be used in the continuity equation. This approximation implicitly assumes that the distribution over u and v is statistically uniform at all points x_0 within the coarse-grained volume at x. This can be abstracted to the case where ΔV is infinitessimally small with respect to the volume of integration. Then the equation can be considered to be true for any volume, and we can pass to the differential form.

In particular, from (**??**) we obtain the following continuity equations,

$$\partial_{\nu}\langle\tilde{T}^{\mu\nu}\rangle+\Gamma^{\mu}_{\sigma\nu}\langle\tilde{T}^{\sigma\nu}\rangle=0 \tag{2.47}$$

$$\partial_{\nu}\langle\tilde{F}^{\mu\nu}\rangle=0 \tag{2.48}$$

where $\tilde{T}$ and $\tilde{F}$ are defined by (4.22) and (3.16) respectively. Note that as in (2.35), (2.47) may instead be written as a covariant derivative of $a^{-4}\langle\tilde{T}^{\mu\nu}\rangle$. Furthermore, since $\langle\tilde{F}^{\mu\nu}\rangle$ is antisymmetric, (2.48) may also be written in terms of a covariant derivative, $\nabla_{\nu}\langle\tilde{F}^{\mu\nu}\rangle=0$.

Evaluating the connection coefficients in (2.47) explicitly using (2.36), we find the energy continuity equation:

$$\begin{aligned}\partial_{\nu}\langle\tilde{T}^{0\nu}\rangle&=-\mathcal{H}\sum_{\sigma}\langle\tilde{T}^{\sigma\sigma}\rangle\\&=-\mathcal{H}\langle\epsilon(1+(\mathbf{v}^2-\mathbf{u}^2))\rangle\\&=-2\mathcal{H}\langle\epsilon\mathbf{v}^2\rangle\end{aligned} \tag{2.49}$$

And following (2.40) we have the momentum continuity equation:

$$\partial_\nu \langle \tilde{T}^{i\nu} \rangle = -2\mathcal{H} \langle \epsilon v^i \rangle \tag{2.50}$$

Also note that the top row of (2.48) does not involve a time derivative, and expresses the differential form of (2.28):

$$\partial_i \langle \epsilon u^i \rangle = 0. \tag{2.51}$$

The continuity equations (2.47) and (2.48) express the time derivatives of the fields $\langle v^i \rangle$ and $\langle u^i \rangle$ in terms of spatial derivatives of correlations such as $\langle u^i u^j \rangle$. In general, these correlations are not factorizable into $\langle v^i \rangle$ and $\langle u^i \rangle$. To simplify the equations further, we will begin by considering a slightly different set of fields.

A solution to the equation of motion in flat space (2.12) for a single string can be expressed in terms of two waves moving in opposite directions

$$x^i(\tau, \sigma) = \frac{a^i(\sigma - \tau) + b^i(\sigma + \tau)}{2}. \tag{2.52}$$

Then it is convenient to consider the quantities $A^i \equiv \partial a^i / \partial \tau$ and $B^i \equiv \partial b^i / \partial \tau$ which can be expressed in terms of u and v:

$$\begin{aligned} A^i &= v^i - u^i \\ B^i &= v^i + u^i. \end{aligned} \tag{2.53}$$

The gauge condition (2.10) implies both $\mathbf{A}$ and $\mathbf{B}$ are unit three-vectors. We can also extend the definitions of A^i and B^i to four-vectors with a timelike component of $+1$.

Although (2.52) does not hold in Friedmann space, we can still define $\mathbf{A}$ and $\mathbf{B}$ using (2.53). The dynamics of $\mathbf{A}$ and $\mathbf{B}$ simplifies along certain paths on the world sheet [46]. These paths may be thought of as generalizations of the paths of constant phase $\sigma \mp \tau$ in (2.52). In both cases, the path associated with $\mathbf{A}$ points in the spatial direction $\mathbf{B}$, and vice-versa.

Note that the symmetric and antisymmetric parts of the tensor product $B \otimes A$ are just $\tilde{T}^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$, respectively,

$$\begin{aligned} \langle \tilde{T}^{\mu\nu} \rangle &= \langle \epsilon B^{(\mu} A^{\nu)} \rangle \\ \langle \tilde{F}^{\mu\nu} \rangle &= \langle \epsilon B^{[\mu} A^{\nu]} \rangle. \end{aligned} \tag{2.54}$$

Then we can rewrite the continuity equations (2.47) and (2.48) in terms of A and B fields as

$$\frac{\partial}{\partial\tau}\langle\epsilon A^i\rangle + \frac{\partial}{\partial x^j}\langle\epsilon A^i B^j\rangle + = -\mathcal{H}\langle\epsilon(A^i + B^i)\rangle \tag{2.55}$$

$$\frac{\partial}{\partial\tau}\langle\epsilon B^i\rangle + \frac{\partial}{\partial x^j}\langle\epsilon B^i A^j\rangle = -\mathcal{H}\langle\epsilon(A^i + B^i)\rangle \tag{2.56}$$

By comparison with (2.50), these suggest a picture in which A^i is a conserved charge moving with velocity B^j, and vice-versa. Indeed just such a relationship is suggested by the paths of constant phase, as discussed above.

2.4.2 Local Equilibrium

We can use the new variables in the argument of the energy-density $f(x, A, B)$. The energy-density function involves many small segments of strings in a given coarse-grained region of space and these segments may interact through reconnections or (if they happen to lie on the same string) through the Nambu-Goto dynamics [9]. By modeling these interactions as an exchange of A and B vectors, a transport equation for $f(x, A, B)$ may be derived. If $f(A, B)$ is homogenous in space, it has been shown [9] that an equilibrium distribution $\partial f_{eq}/\partial\tau = 0$ may be factored into parts depending only on A and B separately,

$$f_{eq}(A, B) \sim f_A(A)\, f_B(B). \tag{2.57}$$

We can treat f as probability distribution, defining the normalized expection value in terms of the coarse-graining brackets (2.46),

$$\bar{Q} \equiv \rho^{-1}\langle\epsilon Q\rangle, \tag{2.58}$$

where the energy density ρ is the normalization factor,

$$\rho \equiv \int f(A, B) dA\, dB. \tag{2.59}$$

Then (2.57) implies that at equilibrium A^i and B^j are independent random variables:

$$\langle\epsilon A^i B^j\rangle = \rho\, \bar{A}^i \bar{B}^j. \tag{2.60}$$

In the general case where f varies in space, we will likewise take ‘local equillibrium’ to mean that A^i and B^j are independent at each point of space.

On the other hand, u^i and v^j are not in general independent, but using (2.54) we can still factor both $T^{\mu\nu}$ and $F^{\mu\nu}$ into $\bar{u}^i$ and $\bar{v}^i$,

$$\langle T^{\mu\nu}\rangle = \rho(\bar{v}^\mu \bar{v}^\nu - \bar{u}^\mu \bar{u}^\nu) \tag{2.61}$$

$$\langle F^{\mu\nu}\rangle = \rho(\bar{u}^\mu \bar{v}^\nu - \bar{v}^\mu \bar{u}^\nu). \tag{2.62}$$

Because $\mathbf{A}$ and $\mathbf{B}$ are unit vectors, the variance does not depend on higher order moments:

$$\begin{aligned}\mathrm{Var}(\mathbf{A}) &= \overline{\mathbf{A}^2} - \bar{\mathbf{A}}^2 \\ &= 1 - \bar{\mathbf{A}}^2 \\ \mathrm{Var}(\mathbf{B}) &= 1 - \bar{\mathbf{B}}^2. \end{aligned} \tag{2.63}$$

And since $\mathbf{u}$ and $\mathbf{v}$ are linear combinations of the independent $\mathbf{A}$ and $\mathbf{B}$,

$$\mathrm{Var}(\mathbf{v}) = \frac{1}{4}(\mathrm{Var}(\mathbf{A}) + \mathrm{Var}(\mathbf{B})) = \mathrm{Var}(\mathbf{u}). \tag{2.64}$$

This can be expressed solely in terms of $\mathbf{u}$ and $\mathbf{v}$ using the gauge condition (2.6):

$$\begin{aligned}\mathrm{Var}(\mathbf{u}) = \mathrm{Var}(\mathbf{v}) &= \frac{1}{4}(2 - (\bar{\mathbf{A}}^2 + \bar{\mathbf{B}}^2)) \\ &= \frac{1}{2}(1 - (\bar{\mathbf{u}}^2 + \bar{\mathbf{v}}^2)). \end{aligned} \tag{2.65}$$

So the variance of $\mathbf{u}$ and $\mathbf{v}$ is related to the extent to which the gauge condition (2.10) is violated by the averaged fields. Likewise, the condition (2.6) is violated whenever $\mathrm{Var}(\mathbf{A}) \neq \mathrm{Var}(\mathbf{B})$. Using (2.63), it is easy to show,

$$\mathrm{Var}(\mathbf{A}) - \mathrm{Var}(\mathbf{B}) = \frac{1}{4}(\bar{\mathbf{v}} \cdot \bar{\mathbf{u}}). \tag{2.66}$$

These expressions involving second order moments are useful in dealing with the factor of $\langle \epsilon \mathbf{v}^2 \rangle$ in the gravitational correction to the energy continuity equation (2.49). From (2.65),

$$\overline{\mathbf{v}^2} = \frac{1}{2}(1 + (\bar{\mathbf{v}}^2 - \bar{\mathbf{u}}^2)). \tag{2.67}$$

2.4.3 Fluid Equations

The continuity equations can now be put in the familiar form of fluid mechanics. Ignoring the gravitational terms for now, we can write (2.47) as the two equations,

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x^j}(\rho \bar{v}^j) = 0 \tag{2.68}$$

and

$$\rho \left(\frac{\partial \bar{v}^i}{\partial \tau} + \bar{v}^j \frac{\partial \bar{v}^i}{\partial x^j} \right) = \frac{\partial \sigma^{ij}}{\partial x^j}. \tag{2.69}$$

where the Cauchy stress tensor is defined as $\sigma^{ij} \equiv \rho\, \bar{u}^i \bar{u}^j$. The stress tensor can be decomposed into a scalar 'pressure',

$$p \equiv -\frac{1}{3}\mathrm{Tr}(\sigma) \tag{2.70}$$

and a traceless 'viscous stress tensor'

$$\epsilon^{ij} \equiv \sigma^{ij} + p\, \delta^{ij}. \tag{2.71}$$

With these definitions we can put (2.69) into the general form of the Navier-Stokes equations,

$$\rho \frac{D\bar{v}^i}{D\tau} = -\frac{\partial p}{\partial x^i} + \frac{\partial \epsilon^{ij}}{\partial x^j} \tag{2.72}$$

where the material derivative

$$\frac{D}{D\tau} \equiv \frac{\partial}{\partial \tau} + \bar{\mathbf{v}} \cdot \nabla. \tag{2.73}$$

We stress, however, that (2.72) differ from the proper Navier-Stokes equations in that the viscous stress tensor ϵ^{ij} can not be written in terms of spatial derivatives of v times a viscosity coefficient.

Although p formally acts like the pressure, it is not clear whether it can be identified with the thermodynamic pressure. If there is a distinction, the viscous stress tensor may be defined with a nonzero trace in which case there would be a non-vanishing bulk viscosity [20]. Also note that the energy-momentum tensor $\rho(\bar{v}^\mu \bar{v}^\nu - \bar{u}^\mu \bar{u}^\nu)$ is not in the form of a perfect fluid. But the condition that ϵ^{ij} vanishes implies that $-\rho\, \bar{u}^i \bar{u}^j = p\delta^{ij}$. This condition is just what is needed to put the energy-momentum tensor in the form of a perfect fluid with pressure p. So p is consistent with the pressure as defined in familiar cosmological models.

In general, it is a lot more informative to rewrite the hydrodynamic equations with a dynamical vector field $\mathbf{u}$ rather than the pressure and viscous tensor. Using (2.51) to simplify (2.49),(2.50), and (2.48), we find,

$$\frac{\partial \rho}{\partial \tau} + \nabla \cdot (\rho \bar{\mathbf{v}}) = -\mathcal{H}\rho\,(1 + \bar{\mathbf{v}}^2 - \bar{\mathbf{u}}^2) \tag{2.74}$$

$$\nabla \cdot (\rho \bar{\mathbf{u}}) = 0 \tag{2.75}$$

$$\frac{D\bar{\mathbf{v}}}{D\tau} - \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = -2\mathcal{H}\bar{\mathbf{v}}. \tag{2.76}$$

$$\frac{D\bar{\mathbf{u}}}{D\tau} - \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{v}} = 0 \tag{2.77}$$

Here (2.67) was used to simplify $\overline{\mathbf{v}^2}$ in the energy continuity equation (2.74). Note that the evolution of the $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ fields decouple from the energy density ρ.

We can also rewrite the decoupled equations (2.77) and (2.76) in terms of the $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ fields using (2.53),

$$\frac{\partial \bar{\mathbf{A}}}{\partial \tau} + \bar{\mathbf{B}} \cdot \nabla \bar{\mathbf{A}} = -\mathcal{H}(\bar{\mathbf{A}} + \bar{\mathbf{B}}) \tag{2.78}$$

$$\frac{\partial \bar{\mathbf{B}}}{\partial \tau} + \bar{\mathbf{A}} \cdot \nabla \bar{\mathbf{B}} = -\mathcal{H}(\bar{\mathbf{A}} + \bar{\mathbf{B}}). \tag{2.79}$$

As discussed in Section 2.4.1, the quantity $\bar{\mathbf{A}}$ can be considered to move with velocity $\bar{\mathbf{B}}$ (and vice-versa) along a path of constant phase. In this respect, the left hand side of equations (2.78) and (2.79) can be interpreted as a material derivative. In flat spacetime, $\mathbf{A}$ and $\mathbf{B}$ are constant along a path of constant phase, and their respective material derivatives vanish. This is an intuitive, but non-trivial result given that the quantities appearing in (2.78) and (2.79) are the local averages of the $\bar{A}$ and $\bar{B}$ values on an individual string. In fact there is no reason to expect that the same equations would describe more general fluids in which the local equilibrium assumption is violated.

Chapter 3

Fluid of Nambu-Goto strings II

3.1 Synopsis

This chapter is taken from the body of the paper 'String Fluid in Local Equilibrium.' [12] The chapter is organized as follows. In Sec. 3.2 we discuss individual Nambu-Goto strings in a manner which makes clear in a coordinate and gauge independent way how the conserved F tensor arises for individual strings. In Sec. 3.3 we develop the coarse-grained equations for a fluid of Nambu-Goto strings, which is here called the 'local equilibrium model.' These equations are equivalent to those in Chapter 2, although they are now derived in a covariant and arguably more direct way.

In section 3.3.1 the Frobenius theorem is used to show that the string fluid is foliated by two-dimensional manifolds similar to macroscopic string worldsheets. And in section 3.3.2, it is shown that if the variance in the local equilibrium model vanishes these macroscopic strings obey the Nambu-Goto equations of motion, and the model reduces to the Stachel model [32].

In Sec. 3.4 we begin analyzing the general case in which the variance does not vanish by considering the eigenvalues of the energy-momentum tensor. In the generic case where the eigenspace is not degenerate we may reformulate the fluid equations in terms of the eigenvectors V and U of the energy-momentum tensor. In section 3.4.1 these new variables are used to show that the submanifolds obey the wiggly string equations of motion. In the remaining case in which the eigenspace is degenerate, it is shown in section 3.4.2 that the submanifolds obey the equations of chiral strings.

3.2 Nambu-Goto Strings

We start by reviewing the basic properties of the individual Nambu-Goto strings. Consider a world-sheet of a single string described by coordinates η^a, where $a = 0, 1$, embedded into the four-dimensional target space $X^\mu(\eta^a)$, where $\mu = 0, 1, 2, 3$. Then we can define a pullback of the target space metric (or the induced metric)

$$h_{ab} \equiv g_{\mu\nu} X^\mu_{,a} X^\nu_{,b}. \tag{3.1}$$

For the Nambu-Goto strings the equations of motions are obtained from the action,

$$S = -\int d^2\eta \sqrt{-h}, \tag{3.2}$$

where the units are chosen to set the string tension coefficient to one, and the corresponding (singular) energy-momentum tensor as a function of the target space coordinates x^λ is given by,

$$T^{\mu\nu}\sqrt{-g} = \int d^2\eta \sqrt{-h}\, h^{ab} X^\mu_{,a} X^\nu_{,b}\, \delta^{(4)}(x^\lambda - X^\lambda). \tag{3.3}$$

Due to conservations of energy and momenta, the energy-momentum tensor should also obey the conservation equation,

$$\nabla_\mu T^{\mu\nu} = 0, \tag{3.4}$$

but because of the presence of the delta function in (4.22), the interpretation of the expression (3.4) is somewhat obscure.

3.2.1 Conservation Equations

To clarify the conservation law (3.4) for a singular energy momentum tensor (4.22), consider first a general singular current of the form

$$J^\mu\sqrt{-g} = \int d\eta^0 \wedge d\eta^1\, \tilde{J}^\mu(\eta)\, \delta^{(4)}(x^\lambda - X^\lambda). \tag{3.5}$$

Then the conserved current J^μ formally obeys the conservation condition in the target space

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(J^\mu\sqrt{-g}) = 0 \tag{3.6}$$

or

$$\partial_\mu (J^\mu \sqrt{-g}) = 0. \tag{3.7}$$

By integrating over a four-dimensional volume, this was shown [11] to imply a conservation condition on the worldsheet

$$\partial_a \tilde{J}^a = 0 \tag{3.8}$$

for a vector $\tilde{J}^a$ which can be pushforward to the conserved current $\tilde{J}^\mu$ in the target space,

$$\tilde{J}^\mu = \tilde{J}^a X^\mu_{,a}. \tag{3.9}$$

(See Ref. [11] for details).

The same procedure can be applied directly to the energy-momentum tensor (4.22) of a Nambu-Goto string in flat space-time. Then the four conservation equations (3.4) in the target space can be put to the same form as (3.7),

$$\partial_\mu (T^{\mu\nu} \sqrt{-g}) = 0 \tag{3.10}$$

and by inspecting (4.22) we can we can identify the four conserved currents on the worldsheet as the four coefficients of $X^\mu_{,a}$ leading to the four familiar equations of motion for Nambu-Goto strings in flat space-time,

$$\partial_a (\sqrt{-h}\, h^{ab} X^\nu_{,b}) = 0. \tag{3.11}$$

For a general space-time metric the equivalent of (3.10) is not true for the second rank tensor $T^{\mu\nu}$ since there is an additional term involving a connection coefficient in the target space conservation equation (3.4),

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + \Gamma^\nu_{\lambda\mu} T^{\lambda\mu} = 0. \tag{3.12}$$

But this simply leads to an additional term $\Gamma^\nu_{\lambda\mu} \tilde{T}^{\lambda\mu}$ in the singular current conservation equation, which can also be pushed-forward to become the Nambu-Goto equation of motion in a general space-time,

$$X^\mu_{,a} \nabla_\mu (\sqrt{-h}\, h^{ab} X^\nu_{,b}) = 0. \tag{3.13}$$

Besides the worldsheet currents associated to the energy-momentum tensor, we can consider a trivial current conservation due to commutation of partial derivatives:

$$\partial_a(\epsilon^{ab}X^{\nu}_{,b}) = 0. \tag{3.14}$$

where ϵ^{ab} is the antisymmetric Levi-Civita symbol. Following the above discussion, this leads to four more conserved currents

$$\nabla_\mu F^{\mu\nu} = 0 \tag{3.15}$$

described by a spacetime tensor,

$$F^{\mu\nu}\sqrt{-g} \equiv \int d\eta^0 \wedge d\eta^1\, \epsilon^{ab} X^{\nu}_{,b} X^{\mu}_{,a}\, \delta^{(4)}(x^\lambda - X^\lambda). \tag{3.16}$$

The conservation of $F^{\mu\nu}$ is related to the continuity of closed or infinite strings at each point and does not depend on a particular choice of the string action such as the Nambu-Goto action [11]. More generally, in models with open strings (which can have endpoints on monopoles or higher dimensional branes) the conservation equations (3.15) may include a source term, but the basic form of the equations would not be expected to change.

3.2.2 Right and Left Movers

In a particular choice of gauge, similarities between $T^{\mu\nu}$ and $F^{\mu\nu}$ become apparent. We will denote the two-forms in the integrands of the expressions (4.22) and (3.16) with a hat,

$$\hat{T}^{\mu\nu} \equiv h^{ab}\sqrt{-h}\, X^{\mu}_{,a} X^{\nu}_{,b}\, d\eta^0 \wedge d\eta^1 \tag{3.17}$$

$$\hat{F}^{\mu\nu} \equiv \epsilon^{ab} X^{\mu}_{,a} X^{\nu}_{,b}\, d\eta^0 \wedge d\eta^1 = dX^\mu \wedge dX^\nu \tag{3.18}$$

To simplify the factor $\sqrt{-h}\, h^{ab}$ in (3.17) we choose η^0 and η^1 to be (left-pointing and right-pointing) conformal lightcone coordinates. In this gauge, the equations (3.17) and (3.18) become,

$$\hat{T}^{\mu\nu} = 2\mathcal{A}^{(\mu}\mathcal{B}^{\nu)} d\eta^0 \wedge d\eta^1 \tag{3.19}$$

$$\hat{F}^{\mu\nu} = 2\mathcal{A}^{[\mu}\mathcal{B}^{\nu]} d\eta^0 \wedge d\eta^1. \tag{3.20}$$

where the two coordinate basis vectors are denoted as

$$\mathcal{A}^\mu \equiv \frac{\partial X^\mu}{\partial \eta^0} \tag{3.21}$$

$$\mathcal{B}^\mu \equiv \frac{\partial X^\mu}{\partial \eta^1}. \tag{3.22}$$

Besides pointing in the two null directions on the worldsheet, $\mathcal{A}^\mu$ and $\mathcal{B}^\mu$ are relevant as the two propagation directions of extrinsic perturbations. But it is only the direction which is physically relevant —there is still some gauge freedom in the normalization. We will define the new vectors A^μ and B^μ normalized to have a unit time component, i.e.

$$\begin{aligned} A^\mu &= \frac{\mathcal{A}^\mu}{\mathcal{A}^0} \\ B^\mu &= \frac{\mathcal{B}^\mu}{\mathcal{B}^0} \end{aligned} \tag{3.23}$$

and expressions (3.19) and (3.20) can be re-written as,

$$\begin{aligned} \hat{T}^{\mu\nu} &= A^{(\mu} B^{\nu)}\, \hat{T}^{00} \\ \hat{F}^{\mu\nu} &= A^{[\mu} B^{\nu]}\, \hat{T}^{00}. \end{aligned} \tag{3.24}$$

We can also define the full spacetime tensor,

$$\begin{aligned} (A \otimes B)^{\mu\nu}(x^\lambda) &\equiv T^{\mu\nu}(x^\lambda) + F^{\mu\nu}(x^\lambda) \\ &= \frac{1}{\sqrt{-g(x^\lambda)}} \int \hat{T}^{00}\, A^\mu B^\nu\, \delta(x^\lambda - X^\lambda(\eta)), \end{aligned} \tag{3.25}$$

which must satisfy,

$$\nabla_\mu (A \otimes B)^{\mu\nu} = \nabla_\nu (A \otimes B)^{\mu\nu} = 0, \tag{3.26}$$

due to the conservation equations (3.4) and (3.15). The string network can also be generalized to contain non-Nambu-Goto strings, and in these cases A^μ and B^μ will be defined as the physical propagation directions rather than the null directions. In particular, the form of $\hat{T}^{\mu\nu}$ and $\hat{F}^{\mu\nu}$ for chiral strings and wiggly strings is identical to the Nambu-Goto case. The only distinction is that one or both of A^μ and B^μ are timelike vectors rather than null vectors [16, 15].

Although the quantities A^μ, B^μ, and T^{00} in (3.25) can be defined independently of the choice of gauge, the price we pay is the loss of manifest spacetime covariance. But since $T^{\mu\nu}$, $F^{\mu\nu}$ and, thus, $(A \otimes B)^{\mu\nu}$ all transform as second rank tensors, the null vectors A^μ and B^μ are uniquely determined in each frame even if their transformation laws are not those of four-vectors. Instead of $\hat{T}^{00}$ it may seem more natural to consider a fully covariant measure such as $\hat{T}^\mu{}_\mu$. According to (3.19), this is also proportional to the worldsheet area

$$\hat{T}^\mu{}_\mu = 2\sqrt{-h}\, d\eta_0 \wedge \eta_1. \tag{3.27}$$

But this measure can be recovered from the quantities A^μ, B^μ, and $\hat{T}^{00}$ through (3.24), and will not be as useful in considering the coarse-grained dynamics.

3.3 Fluid Equations

To develop a fluid description of strings we consider the singular tensor currents $(A \otimes B)^{\mu\nu}$ of all strings in a local neighborhood around each space-time point, x^λ. The coarse-grained currents are then determined by integrating the singular currents over a spacetime volume ΔVabout x^λ,[1]

$$\langle A \otimes B\rangle^{\mu\nu}(x^\lambda) \equiv \frac{1}{\Delta V}\int_{\Delta V} d^4x\,(A \otimes B)^{\mu\nu}. \tag{3.28}$$

Using (3.25) the integral in (3.28) can be calculated by integrating over different pieces of world-sheets enclosed in the volume ΔV with the energy density $\hat{T}^{00}$ as a measure of integration. Then expectation values of the A^μ and B^μ vectors (denoted with a bar) are given by

$$\bar{A}^\mu = \frac{1}{\rho}\langle A \otimes B\rangle^{\mu 0} \tag{3.29}$$

$$\bar{B}^\nu = \frac{1}{\rho}\langle A \otimes B\rangle^{0\nu} \tag{3.30}$$

where

$$\rho \equiv \langle A \otimes B\rangle^{00} \tag{3.31}$$

[1] As usual, the fluid approximation relies on the assumption that the coarse-grained fields do not depend significantly on the choice of ΔV as long as it is from an appropriate range of scales.

is the coarse-grained energy density.

Since the spatial components of the string network quantities A^i and B^i lie on a unit two-sphere (known as the Kibble-Turok sphere), the variances of the averaged fields $\bar{A}^\mu$ and $\bar{B}^\nu$ satisfy simple expressions:

$$\mathrm{Var}(\bar{A}) = \overline{(A^i A_i)} - \bar{A}^i \bar{A}_i = \bar{A}^\mu \bar{A}_\mu \tag{3.32}$$

$$\mathrm{Var}(\bar{B}) = \bar{B}^\mu \bar{B}_\mu. \tag{3.33}$$

Because of this we will refer to the squares of the four-vector magnitudes of $\bar{A}^\mu$ and $\bar{B}^\mu$ as the variances of A^μ and B^μ.

We can now impose the microscopic conservation equations (3.26) to derive macroscopic equations for the coarse-grained field

$$\nabla_\mu \langle A \otimes B \rangle^{(\mu\nu)} = 0 \tag{3.34}$$

and

$$\nabla_\mu \langle A \otimes B \rangle^{[\mu\nu]} = 0. \tag{3.35}$$

These equations are generically underdetermined which can be seen by counting the degrees of freedom. A general second rank tensor $\langle A \otimes B \rangle^{\mu\nu}$ has 16 independent components, but there are only 4 dynamical equations in (3.34) and 3 dynamical (corresponding to $\nu = 1, 2, 3$) and 1 constraint (corresponding to $\nu = 0$) equation in (3.35). This means that the set of equations can only be solved if we reduce the total number of independent components in $\langle A \otimes B \rangle^{\mu\nu}$ to $4 + 3 - 1 = 6$.

To constrain the underdetermined conservation equations (3.34) and (3.35) we will use the further assumption that A^μ and B^μ are statistically independent under the energy-density measure of integration as in equation (3.28). Earlier work on a kinetic theory for string networks indicates that under certain conditions the measure will indeed converge to an equilibrium distribution in which A^μ and B^μ are independent random variables [9]. Throughout paper we will adopt this *local equilibrium* assumption under which

$$\langle A \otimes B \rangle^{\mu\nu} = \rho \bar{A}^\mu \bar{B}^\nu \tag{3.36}$$

and in the last section we will comment on a possible generalization of the string fluid to include the effects of pressure and viscosity which are expected to be important for the fluids of, for example, cosmic strings. In the equilibrium fluid the coarse-grained tensors (3.24) become

$$\langle T\rangle^{\mu\nu} = \rho\bar{A}^{(\mu}\bar{B}^{\nu)} \tag{3.37}$$

$$\langle F\rangle^{\mu\nu} = \rho\bar{A}^{[\mu}\bar{B}^{\nu]}, \tag{3.38}$$

and the conservation equations (3.34) and (3.35) are greatly simplified [11],

$$\nabla_\mu(\rho\bar{A}^\mu\bar{B}^\nu) = 0 \tag{3.39}$$

$$\nabla_\nu(\rho\bar{A}^\mu\bar{B}^\nu) = 0. \tag{3.40}$$

Then the number of independent components is exactly 6 described by the components of the three-vectors A^i and B^i. As we shall argue below the corresponding equations for A^i and B^i are completely decoupled from the equations for the energy density, ρ which is no longer an independent degree of freedom. Once the space-time solutions for A^i and B^i are obtained, the energy density ρ is uniquely determined from certain boundary conditions.

3.3.1 Submanifold Structure

As was already mentioned in the last section, the full tensor $(A\otimes B)^{\mu\nu}$ is a covariant second rank tensor, but A^μ and B^μ do not transform as four-vectors under general coordinate transformations. Similarly, the coarse-grained tensor $\langle A\otimes B\rangle^{\mu\nu}$ is covariant but the individual quantities ρ, $\bar{A}^\mu$ and $\bar{B}^\mu$ appear to depend on the coarse-graining frame. It is valid to simply take these quantities to transform covariantly, but then in a transformed frame they will no longer have a simple interpretation as coarse-grained quantities. For instance, if we take ρ to transform as a scalar, in a new frame it will no longer equal to the energy density, which transforms as a component of a tensor. For the moment, we will take this approach. Later on we will renormalize these quantities in a more manifestly covariant way.

Given these considerations, it is valid to use the product rule to expand (3.39):

$$\nabla_\mu(\rho\bar{A}^\mu\bar{B}^\nu) = \bar{B}^\nu\nabla_\mu(\rho\bar{A}^\mu) + \rho\bar{A}^\mu\partial_\mu\bar{B}^\nu + \rho\bar{A}^\mu\Gamma^\nu_{\mu\lambda}\bar{B}^\lambda = 0 \tag{3.41}$$

but since $\bar{A}^0 = \bar{B}^0 = 1$, the $\nu = 0$ component of equation (3.41) leads to,

$$\nabla_\mu(\rho\bar{A}^\mu) = -\rho\Gamma^0_{\mu\lambda}\bar{A}^\mu\bar{B}^\lambda. \tag{3.42}$$

and by substituting (3.42) back into (3.41),

$$\bar{A}^{\mu}\partial_{\mu}\bar{B}^{\nu} = -\Gamma^{\nu}_{\mu\lambda}\bar{A}^{\mu}\bar{B}^{\lambda} + \Gamma^{0}_{\mu\lambda}\bar{A}^{\mu}\bar{B}^{\lambda}\bar{B}^{\nu}. \tag{3.43}$$

Similarly beginning from (3.40) we get,

$$\nabla_{\mu}(\rho\bar{B}^{\mu}) = -\rho\Gamma^{0}_{\mu\lambda}\bar{B}^{\mu}\bar{A}^{\lambda}. \tag{3.44}$$

and

$$\bar{B}^{\mu}\partial_{\mu}\bar{A}^{\nu} = -\Gamma^{\nu}_{\mu\lambda}\bar{B}^{\mu}\bar{A}^{\lambda} + \Gamma^{0}_{\mu\lambda}\bar{B}^{\mu}\bar{A}^{\lambda}\bar{A}^{\nu}. \tag{3.45}$$

In total we get the four equations (3.42),(3.43),(3.44), and (3.45) which can be written as

$$\nabla_{\mu}(\rho\bar{A}^{\mu}) = -\rho\,\Gamma^{0}_{\kappa\lambda}\bar{A}^{\kappa}\bar{B}^{\lambda} \tag{3.46}$$

$$\nabla_{\mu}(\rho\bar{B}^{\mu}) = -\rho\,\Gamma^{0}_{\kappa\lambda}\bar{A}^{\kappa}\bar{B}^{\lambda} \tag{3.47}$$

$$\bar{A}^{\mu}\nabla_{\mu}\bar{B}^{\nu} = \Gamma^{0}_{\kappa\lambda}\bar{A}^{\kappa}\bar{B}^{\lambda}\,\bar{B}^{\nu} \tag{3.48}$$

$$\bar{B}^{\mu}\nabla_{\mu}\bar{A}^{\nu} = \Gamma^{0}_{\kappa\lambda}\bar{A}^{\kappa}\bar{B}^{\lambda}\,\bar{A}^{\nu}. \tag{3.49}$$

In particular equations (3.48) and (3.49) imply that the commutator

$$\begin{aligned}[\bar{A},\bar{B}]^{\nu} &\equiv \bar{A}^{\mu}\nabla_{\mu}\bar{B}^{\nu} - \bar{B}^{\mu}\nabla_{\mu}\bar{A}^{\nu} \\ &= \Gamma^{0}_{\kappa\lambda}\bar{A}^{\kappa}\bar{B}^{\lambda}\,(\bar{B}^{\nu} - \bar{A}^{\nu})\end{aligned} \tag{3.50}$$

lies everywhere in the space spanned by $\bar{A}^{\mu}$ and $\bar{B}^{\mu}$. Thus by Frobenius' theorem, spacetime can be foliated by a family of two-dimensional submanifolds everywhere tangent to $\bar{A}^{\mu}$ and $\bar{B}^{\mu}$. These submanifolds may be thought of as the worldsheets of the one-dimensional field lines of the spacelike vector field $\bar{B}^{\mu} - \bar{A}^{\mu}$, which is nothing but the vector field describing the average local direction (or tangent vector) of strings.

These submanifolds clarify the Cauchy problem for the string fluid in local equilibrium. If $\bar{A}^{\mu}$ and $\bar{B}^{\mu}$ are specified on a field line at an initial time, equations (3.48) and (3.49) can be used to solve for the values of $\bar{A}^{\mu}$ and $\bar{B}^{\mu}$ along the full submanifold. The possibility of the intersection of submanifolds physically indicates shockwaves which are not resolved in the equilibrium fluid [11]. But if $\bar{A}^{\mu}$ and $\bar{B}^{\mu}$ are given as initial conditions then the solution can be propagated forward for at least some finite time. Notice that the solution of equations (3.48) and (3.49) for $\bar{A}^\mu$ and $\bar{B}^\mu$ does not depend on ρ, but using the solution for $\bar{A}^\mu$ and $\bar{B}^\mu$, equations (3.46) and (3.47) determine the full ρ field given the specification of an initial ρ at one point on each submanifold.

This property of forming two-dimensional submanifolds may also hold for a more general string fluid. If the tensor $\langle F\rangle^{\mu\nu}$ annihilates exactly two linearly independent directions, it can be shown that it is a *simple* bivector —that is, there exists two vector fields ξ^μ and ζ^μ such that,

$$\langle F\rangle^{\mu\nu} = \xi^\mu\zeta^\nu - \zeta^\mu\xi^\nu. \tag{3.51}$$

On the other hand, the dual tensor

$$\begin{aligned} \star\langle F\rangle^{\mu\nu} &\equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\langle F\rangle_{\rho\sigma} \\ &= \epsilon^{\mu\nu\rho\sigma}\xi_\rho\zeta_\sigma \end{aligned} \tag{3.52}$$

annihilates vectors in the space spanned by ξ^μ and ζ^μ and, thus, the Frobenius condition for ξ^μ and ζ^μ to form surfaces can be expressed as

$$\star\langle F\rangle_{\mu\nu}\,[\xi,\zeta]^\nu = 0. \tag{3.53}$$

Now if $\langle F\rangle^{\mu\nu}$ is a simple bivector (3.51), then the conservation law

$$\nabla_\mu\langle F\rangle^{\mu\nu} = (\nabla_\lambda\xi^\lambda)\zeta^\nu - (\nabla_\lambda\zeta^\lambda)\xi^\nu + [\xi,\zeta]^\nu = 0 \tag{3.54}$$

which holds for any string fluid can be used to obtain the Frobenius condition (3.53),

$$\star\langle F\rangle_{\mu\nu}\,[\xi,\zeta]^\nu = \star\langle F\rangle_{\mu\nu}\,(-(\nabla_\lambda\xi^\lambda)\zeta^\nu + (\nabla_\lambda\zeta^\lambda)\xi^\nu) = 0. \tag{3.55}$$

Once again we have used the fact that $\star\langle F\rangle^{\mu\nu}$ annihilates vectors ξ^μ and ζ^μ. So under the condition of local equilibrium the fluid is foliated by a collection of submanifolds, each of which independently acts like the worldsheet of a string.

3.3.2 Nambu-Goto String Dust

A similar "string dust" model was introduced by Stachel [32][33] in which each submanifold respects the Nambu-Goto action. In fact, the local equilibrium model is exactly the Stachel model when both $\bar{A}^\mu$ and $\bar{B}^\mu$ are restricted to be linearly independent null vectors. In that case the equations (3.48) and (3.49) are just the equations for a Nambu-Goto string expressed in terms of the vectors A^μ and B^μ defined in (3.23) (see e.g. [10]). Of course if there are no statistical variances, the mean $\bar{A}^\mu$ and $\bar{B}^\mu$ are just equal to the A^μ and B^μ for each individual Nambu-Goto string in the coarse-grained network, so this result would be expected.

The connection to the string dust model is more easily seen in a normalized notation. We can always choose the vectors ξ and ζ forming $\langle F\rangle$ in (3.51) to be orthogonal, and we can also factor out any overall magnitude into a scalar φ so that we are left with a pair of orthonormal vectors —one timelike, v^μ, and one spacelike, u^μ, i.e.[2]

$$v_\mu v^\mu = -u_\mu u^\mu = 1 \tag{3.56}$$

$$u_\mu v^\mu = 0 \tag{3.57}$$

$$\langle F\rangle^{\mu\nu} = \varphi(u^\mu v^\nu - v^\mu u^\nu). \tag{3.58}$$

The unit bivector in parenthesis is denoted as

$$\Sigma^{\mu\nu} \equiv u^\mu v^\nu - v^\mu u^\nu, \tag{3.59}$$

and the quantity φ can be found from the contraction of $\langle F\rangle$

$$\varphi \equiv \sqrt{-\frac{1}{2}\langle F\rangle^{\mu\nu}\langle F\rangle_{\mu\nu}} \tag{3.60}$$

The projector onto the submanifold $h^{\mu\nu}$ can be also defined in terms of the unit simple bivector,

$$h^{\mu\nu} = \Sigma^{\mu\rho}\Sigma_\rho{}^\nu = v^\mu v^\nu - u^\mu u^\nu. \tag{3.61}$$

Note that this is also the pushforward of the inverse metric h^{ab} on the worldsheet, hence the same choice of notation.

For the equilibrium string fluid the bivector magnitude is given by

$$\begin{aligned}\varphi &= \rho\sqrt{-\frac{1}{2}\bar{A}^{[\mu}\bar{B}^{\nu]}\bar{A}_{[\mu}\bar{B}_{\nu]}} \\ &= \frac{\rho}{2}\sqrt{(\bar{A}^\lambda\bar{B}_\lambda)^2 - |\bar{A}|^2|\bar{B}|^2}.\end{aligned} \tag{3.62}$$

[2] Here the convention is that the letter v^μ is taken to be the timelike vector, and u^μ the spacelike vector. This notation is the opposite of the convention in certain papers, but is consistent with the notation in [11, 10].

and if either of the variances of $\bar{A}^\mu$ or $\bar{B}^\mu$ vanish equations (3.32) and (3.33) imply that the magnitude is proportional to the trace of the energy-momentum tensor:

$$\begin{aligned} \varphi &= \frac{\rho}{2}\bar{A}^\lambda \bar{B}_\lambda && (3.63) \\ &= \frac{1}{2}\langle T\rangle^\lambda{}_\lambda. && (3.64) \end{aligned}$$

Then by (3.27) the magnitude φ can also be interpreted as the coarse-grained worldsheet area in the underlying string network (this will not be true when both $\bar{A}^\mu$ and $\bar{B}^\mu$ have statistical variance). Moreover, when both variances vanish, the simple bivector $\langle F\rangle^{\mu\nu}$ itself can be related to $\langle T\rangle^{\mu\nu}$,

$$\begin{aligned} \frac{1}{\varphi}\langle F\rangle^{\mu\lambda}\langle F\rangle_{\lambda\nu} &= \frac{\rho^2}{4\varphi}(\bar{A}^\mu \bar{B}^\lambda \bar{A}_\lambda \bar{B}_\nu + \bar{B}^\mu \bar{B}^\lambda \bar{A}_\lambda \bar{A}_\nu) \\ &= \frac{\rho}{2}(\bar{A}^\mu \bar{B}_\nu + \bar{B}^\mu \bar{A}_\nu) = \langle T\rangle^\mu{}_\nu. \qquad (3.65) \end{aligned}$$

and using (3.61) the energy-momentum tensor may be written in terms of the bivector magnitude, φ, and unit bivector, $\Sigma^{\mu\nu}$,

$$\langle T\rangle^{\mu\nu} = \varphi\,\Sigma^\mu{}_\lambda \Sigma^{\lambda\nu} = \varphi h^{\mu\nu} = \varphi(v^\mu v^\nu - u^\mu u^\nu). \qquad (3.66)$$

This choice of energy-momentum tensor was the starting point for the analysis in Stachel's paper [32]. In our model it is seen as a special case of a coarse-grained network of strings in local equilibrium and under the condition that the statistical variations in both vectors A^μ or B^μ are negligible.

3.4 Equilibrium Fluids

The full local equilibrium model in which there may be non-zero variances is more general than the Stachel model [32]. First consider the degenerate case in which $\bar{A} = \bar{B}$. Then $\langle F\rangle^{\mu\nu}$ vanishes and the energy momentum tensor becomes,

$$\langle T\rangle^{\mu\nu} = \rho \bar{A}^\mu \bar{A}^\nu \qquad (3.67)$$

which is formally equivalent to a dust of particles with four-velocity in the direction of $\bar{A}^\mu$. In terms of the underlying string network, this represents a dust of loops which are smaller than the coarse-graining scale.

To clarify the general case when $\bar{A}^\mu$ and $\bar{B}^\mu$ are linearly independent, we will temporarily make use of a non-coordinate basis in which $\bar{A}$ and $\bar{B}$ are taken as basis vectors. In this basis, $\bar{A}^\mu = (1,0,0,0)$ and $\bar{B}^\mu = (0,1,0,0)$, with the other two directions orthogonal. Now the nontrivial components of $\langle T\rangle^\mu{}_\nu$ in equation (3.65) can be written as the two-dimensional matrix $\mathbb{T}$,

$$\mathbb{T} = \frac{\rho}{2}\begin{pmatrix} \bar{A}^\nu\bar{B}_\nu & |\bar{B}|^2 \\ |\bar{A}|^2 & \bar{A}^\nu\bar{B}_\nu \end{pmatrix}. \tag{3.68}$$

whose eigenvalues λ are solutions of the characteristic equation,

$$\left(\frac{\rho}{2}\bar{A}^\nu\bar{B}_\nu - \lambda\right)^2 - \left(\frac{\rho}{2}\right)^2|\bar{A}|^2|\bar{B}|^2 = 0. \tag{3.69}$$

In a degenerate case when either $|\bar{A}|^2$ or $|\bar{B}|^2$ vanishes the only solution of (3.69) is

$$\lambda = \frac{\rho}{2}\bar{A}^\nu\bar{B}_\nu = \varphi. \tag{3.70}$$

If both variances vanish the eigenspace is indeed degenerate since $\mathbb{T}$ is just φ multiplied by the projector on the space spanned by $\bar{A}^\mu$ and $\bar{B}^\mu$ —this is just what (3.66) indicates. But if for instance $|\bar{A}|^2 = 0$ but $|\bar{B}|^2 \neq 0$, then the null vector $\bar{A}$ is the only independent eigenvector. We will return to this case in Sec. 3.4.2, where it will be seen that the submanifolds obey the equations of a chiral string with a null-current in the direction of $\bar{A}^\mu$.

For now consider the case in which both $\bar{A}^\mu$ and $\bar{B}^\mu$ are timelike vectors. Then it is easy to verify from (3.68) that $(\pm|\bar{A}|^{-1}, |\bar{B}|^{-1})$ are two eigenvectors with eigenvalues $\rho/2(\bar{A}^\nu\bar{B}_\nu \pm |\bar{A}||\bar{B}|)$, respectively. This suggests to renormalize $\bar{A}$ and $\bar{B}$ to have unit magnitude,

$$\alpha^\mu \equiv \frac{\bar{A}^\mu}{|\bar{A}|}$$
$$\beta^\mu \equiv \frac{\bar{B}^\mu}{|\bar{B}|}, \tag{3.71}$$

so that the eigenvectors are a linear combination of α^μ and β^μ,

$$V^\mu \equiv \frac{1}{2}(\beta^\mu + \alpha^\mu) \tag{3.72}$$

$$U^\mu \equiv \frac{1}{2}(\beta^\mu - \alpha^\mu). \tag{3.73}$$

By the reverse Cauchy-Schwartz inequality that holds for timelike vectors, $|\alpha^\nu \beta_\nu| \geq 1$. This implies that V^μ is timelike and U^μ is spacelike. It is also straightforward to show that V^μ and U^μ are orthogonal

$$V^\nu U_\nu = 0 \tag{3.74}$$

and that their magnitudes satisfy a hyperbolic relationship,

$$|V|^2 - |U|^2 = 1. \tag{3.75}$$

3.4.1 Wiggly String Dust

One of the advantages to considering the normalized fields α^μ and β^μ is that they have simple transformation properties. Earlier we were faced with a non-covariant rule of how to transform $\bar{A}^\mu$ and $\bar{B}^\mu$ under coordinate transformations. If these quantities are always defined as average propagation directions in whichever coordinates we are using then they do not transform as four-vectors. This issue can be clarified by rewriting (3.36) in terms of α^μ and β^μ defined in (3.71),

$$\langle A \otimes B \rangle^{\mu\nu} = \rho \bar{A}^\mu \bar{B}^\nu = \rho' \alpha^\mu \beta^\nu. \tag{3.76}$$

where

$$\rho' \equiv \rho\, |\bar{A}|\, |\bar{B}| = \sqrt{\langle A \otimes B \rangle^{\mu\nu} \langle A \otimes B \rangle_{\mu\nu}} \tag{3.77}$$

is a scalar quantity and α^μ and β^μ are the unit four-vectors and thus transform covariantly under coordinate transformations.

In terms of the newly defined quantities the fluid equations (3.39) and (3.40) can be rewritten in manifestly covariant form,

$$\nabla_\mu (\rho' \alpha^\mu \beta^\nu) = 0 \tag{3.78}$$

$$\nabla_\nu (\rho' \alpha^\mu \beta^\nu) = 0. \tag{3.79}$$

As before, we can decouple the equations by contracting (3.78) and (3.79) with β_ν and α_ν respectively. Using the normalization conditions

$$\alpha_\mu \alpha^\mu = \beta_\mu \beta^\mu = 1 \tag{3.80}$$

we recover two equations,

$$\nabla_\mu(\rho'\alpha^\mu) = 0 \tag{3.81}$$

$$\nabla_\mu(\rho'\beta^\mu) = 0 \tag{3.82}$$

which can be substituted back into (3.78) and (3.79) to obtain two more equations

$$\alpha^\mu \nabla_\mu \beta^\nu = 0, \tag{3.83}$$

$$\beta^\mu \nabla_\mu \alpha^\nu = 0. \tag{3.84}$$

Note that equations (3.83) and (3.84) imply that α^μ and β^μ are the basis vectors for some coordinates on the submanifolds since the commutator vanishes

$$[\alpha, \beta]^\nu \equiv \alpha^\mu \nabla_\mu \beta^\nu - \beta^\mu \nabla_\mu \alpha^\nu = 0. \tag{3.85}$$

Moreover the scalar ρ' had completely decoupled from these equations and is determined by equations (3.81) and (3.82).

The equations (3.83) and (3.84) may also be rewritten in terms of the eigenvectors U^μ and V^μ related to α^μ and β^μ through equations (3.72) and (3.73),

$$V^\mu \nabla_\mu U^\nu - U^\mu \nabla_\mu V^\nu = 0 \tag{3.86}$$

$$V^\mu \nabla_\mu V^\nu - U^\mu \nabla_\mu U^\nu = 0. \tag{3.87}$$

The vanishing of the commutator of U and V in (3.86) indicates that U^μ and V^μ are also coordinate basis vectors for some coordinates σ and τ on a submanifold, i.e.

$$\begin{aligned} V^\mu &= \frac{\partial X^\mu}{\partial \tau} \\ U^\mu &= \frac{\partial X^\mu}{\partial \sigma}. \end{aligned} \tag{3.88}$$

Then equation (3.87) can be view as a wave equation for the embedding of the submanifold coordinates in the target space. For example, in flat spacetime equation (3.87) reduces to

$$\frac{\partial^2 X^\mu}{\partial \tau^2} - \frac{\partial^2 X^\mu}{\partial \sigma^2} = 0. \tag{3.89}$$

where in contrast to the Nambu-Goto case the coordinates σ and τ are not necessarily conformal. Instead these equations for the submanifold are equivalent to those of a wiggly string. The wave equation (3.89) appears in terms of timelike $\bar{A}^\mu$ and $\bar{B}^\mu$ in a paper by Vilenkin [15], and the equations (3.83) and (3.84) for α^μ and β^μ appear in a paper by Carter [16].

To further see that the submanifold obeys the wiggly string equation of state, notice that (3.87) can be interpreted as the conservation of a tensor current on the submanifold, much like the Nambu-Goto equation (3.13) was related to the conservation of the energy-momentum tensor (4.22). Similarly to equation (4.22) we can define a conserved but singular energy-momentum tensor

$$T^{\mu\nu}\sqrt{-g} = \int d^2\eta\, \tilde{T}^{\mu\nu}(\eta)\, \delta(x^\lambda - X^\lambda) \tag{3.90}$$

with support on the submanifold, which involves the pushforward of a worldsheet current to the target space,

$$\tilde{T}^{\mu\nu} = V^\mu V^\nu - U^\mu U^\nu. \tag{3.91}$$

The main difference now is that $\tilde{T}^{\mu\nu}$ can be defined for quite general models of strings in terms of the surface energy density M and the surface tension T [37],

$$\tilde{T}^{\mu\nu} = \sqrt{-h}(M v^\mu v^\nu - T u^\mu u^\nu), \tag{3.92}$$

where as before v^μ and u^μ are the unit eigenvectors of the energy-momentum tensor. But in the σ, τ coordinate system the induced metric (3.1) is

$$h_{ab} = \begin{pmatrix} V^\mu V_\mu & V^\mu U_\mu \\ U^\mu V_\mu & U^\mu U_\mu \end{pmatrix} \tag{3.93}$$

and thus

$$\sqrt{-h} = |V|\,|U|. \tag{3.94}$$

Then equations (3.91) and (3.91) imply,

$$M = \frac{|V|}{|U|} \tag{3.95}$$

$$T = \frac{|U|}{|V|} \tag{3.96}$$

and the submanifold indeed obey the wiggly string equation of state [16, 47]:

$$M\,T = 1. \tag{3.97}$$

3.4.2 Chiral String Dust

Now we come back to the remaining case when the statistical variance in only one of the propagating directions vanishes. Without loss of generality we can assume that the corse-grained tensors

$$\langle A \otimes B \rangle^{\mu\nu} = \rho \bar{A}^{\mu} \bar{B}^{\nu} \tag{3.98}$$

where A^{μ} is a null vector (or $|\bar{A}| = 0$) and B^{μ} is a time-like vector (or $|\bar{B}| > 0$). In flat spacetime the equations of motion (3.48) and (3.49) reduce to the wave equation (3.89) with the difference that the spatial part of $\bar{B}$ lies inside of the Kibble-Turok sphere. This is just the equation of motion for a chiral string [48, 49, 50]. We will further show that the submanifolds obey the equations of a chiral string in arbitrary background metric.

We can renormalize ρ to the scalar φ defined by (3.60) and given by (3.63), $\bar{B}^{\mu}$ to a unit vector β^{μ}, and then absorb all of the normalization factors into a new vector n^{μ} in the direction of $\bar{A}^{\mu}$,

$$\varphi = \frac{\rho}{2}(\bar{A}^{\lambda}\bar{B}_{\lambda}) \tag{3.99}$$

$$\beta^{\mu} = \frac{B^{\mu}}{|\bar{B}|} \tag{3.100}$$

$$n^{\mu} \equiv \frac{2|\bar{B}|\bar{A}^{\mu}}{\bar{A}^{\lambda}\bar{B}_{\lambda}} \tag{3.101}$$

so that (3.98) can be written as

$$\langle A \otimes B \rangle^{\mu\nu} = \varphi \beta^{\mu} n^{\nu}. \tag{3.102}$$

Following the Carter and Peter's paper on the chiral string model [48] we can define the other linearly independent null vector,

$$m^{\mu} \equiv \beta^{\mu} - \frac{1}{2} n^{\mu}, \tag{3.103}$$

then

$$m^{\mu} m_{\mu} = \beta^{\mu}\beta_{\mu} - \frac{1}{2}\beta^{\mu} n_{\mu} + \frac{1}{4} n^{\mu} n_{\mu} = 0 \tag{3.104}$$

$$m^{\mu} n_{\mu} = \beta^{\mu} n_{\mu} - \frac{1}{2}\beta^{\mu} n_{\mu} = 1. \tag{3.105}$$

By considering the conservation equations for $\langle A \otimes B\rangle^{\mu\nu}$ in the same manner as before we find:

$$\nabla_\lambda(\varphi n^\lambda) = 0 \tag{3.106}$$

$$n^\lambda \nabla_\lambda \beta^\mu = 0. \tag{3.107}$$

and by contracting (3.107) with $2m_\mu$ we see that n^μ is indeed a conserved null current.

$$\begin{aligned} 2m_\mu n^\lambda \nabla_\lambda \beta^\mu &= m_\mu n^\lambda \nabla_\lambda (2m^\mu + n^\mu) \\ &= (m_\mu n^\lambda + n_\mu m^\lambda)\nabla_\lambda n^\mu \\ &= h^\lambda{}_\mu \nabla_\lambda n^\mu = 0. \end{aligned} \tag{3.108}$$

where h is a projector on the worldsheet as in equation (3.61). Taking the surface energy-momentum tensor $T^{\mu\nu}$ as usual to be $\langle T\rangle^{\mu\nu}$ with φ factored out,

$$T^{\lambda\mu} = n^{(\lambda}\beta^{\mu)} \tag{3.109}$$

$$= n^\lambda n^\mu + n^{(\lambda} m^{\mu)} = n^\lambda n^\mu + h^{\lambda\mu}, \tag{3.110}$$

which again agrees with the chiral string model in [48, 49, 50].

Chapter 4

Generalized fluids of topological defects

4.1 Synopsis

This chapter is taken from the body of the paper 'Field Theory for Perfect String Fluids.' [13] The chapter is organized as follows. In Sec.4.2 we define a perfect string fluid as a generalization of an ordinary perfect fluid with an additional conserved flux. The energy-momentum tensor is derived from a Langrangian as a function of three scalar fields. Since the first draft of this paper it was realized that ideal magnetohydrodynamics is a particular example of a perfect string fluid [14], and the demonstration of this result is reproduced here.

In Sec.4.3 we investigate string fluids for which the pressure vanishes. Examples of this case include the Stachel-Letelier model [32][33] and a recent description of coarse-grained Nambu-Goto strings [12]. It is shown that in these cases the string fluid is foliated by worldsheets of a general form of string described by Brandon Carter [38]. And so these classical strings may be alternatively described using the variational principle of this paper.

In Sec.4.4 we discuss the variational principle in more depth, describing the relabeling symmetries of the fields and the corresponding Noether symmetries. And in Sec.4.6 the relationship of the variational principle to the familiar description of fluids in terms of Clebsch potentials is discussed. By trading one of the previous scalar fields for a Clebsch potential we derive a modified string fluid which is shown to be equivalent to a model of a superfluid by Carter and Langlois [39].

In Sec.4.5 we extend the perfect string fluid to fluids of higher dimensional branes. A particularly simple case of a fluid foliated by domain walls is discussed and shown to be related to the ordinary theory of a massless scalar field.

In Sec.4.7 we extend the perfect string fluid by allowing dependence on additional currents and fluxes such as a conserved entropy density. Two complementary approaches are discussed to achieve this. One approach introduces no extra fields but breaks the relabeling symmetry in the Lagrangian. The other [29] maintains the symmetry by introducing additional fields.

4.2 Perfect string fluid

The energy momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}, \tag{4.1}$$

where u is the unit velocity of the fluid, p is the pressure, and ρ is the energy density in the rest frame of u. The energy density ρ is a function of the number densities n_a indexed by a, and we can form the corresponding chemical potentials

$$\mu^a \equiv \frac{\partial \rho}{\partial n_a}. \tag{4.2}$$

These number densities can be the density of any extensive quantity such as baryon number, charge, or entropy (in which case the chemical potential is the temperature). The pressure p is then defined through the usual thermodynamic relation, which defines it essentially as a Legendre transform of ρ,

$$p = -\rho + \mu^a n_a \tag{4.3}$$

Then using (4.2) we get

$$\begin{aligned} dp &= -d\rho + \mu_a dn^a + n_a d\mu^a \\ &= n_a d\mu^a. \end{aligned} \tag{4.4}$$

In addition to the conservation equation for energy-momentum we have the continuity equations for the currents $n_a^\mu \equiv n_a u^\mu$,

$$\nabla_\mu n_a^\mu = 0. \tag{4.5}$$

By using these continuity equations and (4.4), the conservation of $T^{\mu\nu}$

$$\nabla_\mu[(\mu^a n_a)u^\mu u^\nu - pg^{\mu\nu}] = 0,$$

can be reduced to the following equations of motion:

$$n_a^\mu \nabla_{[\mu}(\mu^a u_{\nu]}) = 0. \tag{4.6}$$

What we are calling a perfect string fluid has in addition to the conserved current n^μ (we will consider only one current for the moment) a conserved bivector F,

$$\nabla_\mu F^{\mu\nu} = 0. \tag{4.7}$$

This bivector can be understood as representing a conserved flux in the system such as angular momentum or magnetic flux. In the magnetic case F is just the dual of the electromagnetic tensor, and for this reason ideal magnetohydrodynamics can be treated as a special case of this perfect string fluid formalism [14]. But more generally for any network of oriented strings there will be a bivector associated with the topological flux of strings.

In the perfect string fluid, F is also constrained to be a *simple* bivector, i.e. it has exactly two linearly independent eigenvectors, and the fluid velocity u is in the linear subspace spanned by these eigenvectors. It is then convenient to define the 'string flux' scalar φ and the normalized bivector Σ as the magnitude and direction of F,

$$F^{\mu\nu} = \varphi\Sigma^{\mu\nu} \tag{4.8}$$

$$\Sigma^{\mu\nu}\Sigma_{\mu\nu} = -2. \tag{4.9}$$

The orthogonal to u spacelike direction w is defined from Σ and u,

$$w^\mu \equiv \Sigma^{\mu\nu}u_\nu, \tag{4.10}$$

in terms of which we can choose to express Σ as,

$$\Sigma^{\mu\nu} = w^\mu u^\nu - u^\mu w^\nu \tag{4.11}$$

$$u^\mu u_\mu = -w^\mu w_\mu = 1 \tag{4.12}$$

$$u^\mu w_\mu = 0 \tag{4.13}$$

The projector h onto the linear subspace spanned by u and w can also be defined in terms of Σ,

$$h^{\mu\nu} \equiv u^\mu u^\nu - w^\mu w^\nu = \Sigma^{\mu\rho}\Sigma_\rho{}^\nu. \tag{4.14}$$

The conservation condition on F (4.7) implies through the Frobenius theorem that u and w lie along two-dimensional integrable submanifolds that can be identified as string worldsheets [12]. And the dual tensor to F, $\tilde{F}$ is a two-form that can be integrated to give the flux of these strings across a surface. The conservation of F just implies that the net flux of strings through any closed surface is zero.

The dual to the current n, which we denote by $\tilde{n}$, will also be useful as it is a three-form that can be integrated over a volume to give the conserved charge contained. These two differential forms, $\tilde{n}$ and $\tilde{F}$, have a natural interpretation in terms of Lagrangian coordinates labeling fluid particles. There is a two-dimensional space of distinct 'world-sheet' submanifolds that we can label with the coordinates X and Y. There is an implicit map that specifies which worldsheet passes through a given spacetime point that allows us to define X and Y as functions on spacetime. The two-dimensional surfaces along which both X and Y take constant values are just the worldsheets. As we will discuss later there is a great deal of symmetry in how we choose these coordinates but we do choose them so that the measure $dX \wedge dY$ is just the string flux. In fact this will be taken as a definition,

$$\tilde{F} \equiv dX \wedge dY, \tag{4.15}$$

and thus we define the dual bivector F in (4.7) ultimately in terms of X and Y fields.

X and Y specify a distinct worldsheet, but to label the distinct fluid particles along the string we need a third coordinate Z. The one-dimensional spaces along which all three coordinates are constant are just the particle worldlines. Again we will fix the measure $dX \wedge dY \wedge dZ$, taking it to be the number density,

$$\tilde{n} \equiv dX \wedge dY \wedge dZ, \tag{4.16}$$

and so the current n^μ and thus the directions u and w (through (4.10)) are also specified in terms of these three scalar fields (the Lagrangian coordinates).

An important thing to note about the use of Lagrangian coordinates is that the continuity equations (4.5) and (4.7) are satisfied by construction,

$$d\tilde{n} = 0 \tag{4.17}$$

$$d\tilde{F} = 0. \tag{4.18}$$

To get a complete set of equations of motion we only need to add the conservation of the energy-momentum tensor which is specified by choosing a Lagrangian as a certain function of the X, Y and Z fields,

$$\mathcal{L} = \mathcal{L}\left(\frac{1}{2}\left(dX \wedge dY\right)^2, -\frac{1}{3!}\left(dX \wedge dY \wedge dZ\right)^2\right). \tag{4.19}$$

Note that the Lagrangian for perfect string fluids, $\mathcal{L}(\varphi, n)$, only depends on the scalar fields through the combinations φ^2 and n^2,

$$\varphi^2 = \frac{1}{2}\tilde{F}^{\lambda\mu}\tilde{F}_{\lambda\mu} = \frac{1}{2}\left(dX \wedge dY\right)^2 \tag{4.20}$$

$$n^2 = -\frac{1}{3!}\tilde{n}^{\lambda\mu\nu}\tilde{n}_{\lambda\mu\nu} = -\frac{1}{3!}\left(dX \wedge dY \wedge dZ\right)^2. \tag{4.21}$$

Varying the Lagrangian by $g_{\mu\nu}$ we find $T^{\mu\nu}$:

$$\begin{aligned} T^{\mu\nu} &= 2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} - \mathcal{L}g^{\mu\nu} \\ &= 2[\frac{\partial\mathcal{L}}{\partial\varphi^2}\varphi^2(g^{\mu\nu} - h^{\mu\nu}) + \frac{\partial\mathcal{L}}{\partial n^2}n^2(g^{\mu\nu} - u^\mu u^\nu)] - \mathcal{L}g^{\mu\nu} \\ &= (\rho + p)u^\mu u^\nu - (\tau + p)w^\mu w^\nu - pg^{\mu\nu} \end{aligned} \tag{4.22}$$

where we define

$$\rho \equiv -\mathcal{L} \tag{4.23}$$

$$p \equiv \mathcal{L} - \mathcal{L}_{,\varphi}\varphi - \mathcal{L}_{,n}n, \tag{4.24}$$

and the new thermodynamic potential τ related to the string tension,

$$\tau \equiv -\mathcal{L} + \mathcal{L}_{,n} n, . \tag{4.25}$$

Energy-momentum tensors of this form have been applied for instance to the study of blackfolds [26] and anisotropic cosmological models [34][35][36]. Our focus here is to study the variational principle underlying this fluid, and show how modifications of the Lagrangian can lead to more general models of interacting fluids.

First of all, if $\mathcal{L}$ does not depend on φ the perfect string fluid reduces to the ordinary perfect fluid. This approach to perfect fluids in terms of a variational principle and Lagrangian coordinates is well established (see [30] for a review). Usually the variational principle is expressed by varying the worldlines in the action through diffeomorphisms. But as we show later, we can also treat X, Y, Z as ordinary scalar fields which can be varied independently to produce the equations of motion. A similar field theory perspective for perfect fluids is found in [29].

A perfect string fluid can also be understood as a generalization of ideal magnetohydrodynamics [14]. In the isentropic case the electric field vanishes in the rest frame of the fluid [18]. In terms of the electromagnetic field tensor $\tilde{F}$ this can be written

$$\tilde{F}^{\mu\nu} u_\nu = 0. \tag{4.26}$$

In the string fluid context this is the condition for F to be a simple bivector, and for u to be in its linear subspace. As discussed previously, this implies that there are two dimensional 'worldsheets' which are everywhere tangent to the velocity and the magnetic field, which is also in the linear subspace of F. But in the context of magnetohydrodynamics this is just the statement that the magnetic field lines are "frozen-in" and dragged along by the velocity of the fluid.

In the standard covariant description of magnetohydrodynamics the energy-momentum tensor is simply the sum of a perfect fluid part and an electromagnetic field part [19]. Any interaction between the two sectors takes place implicitly through the conservation of energy-momentum and in the frozen-in field line condition. Given the variational principle for a perfect fluid discussed above, the Lagrangian for ideal magnetohydrodynamics can be expressed as

$$\mathcal{L} = -\rho_0(n^2) - \frac{1}{2}\varphi^2 = -\rho_0\left(\frac{1}{3!}\tilde{n}^{\rho\sigma\kappa}\tilde{n}_{\rho\sigma\kappa}\right) - \frac{1}{4}\tilde{F}^{\rho\sigma}\tilde{F}_{\rho\sigma} = \tag{4.27}$$

$$= -\rho_0\left(-\frac{1}{3!}(dX \wedge dY \wedge dZ)^2\right) - \frac{1}{4}(dX \wedge dY)^2, \tag{4.28}$$

where $\rho_0(n^2)$ is the energy density of the perfect fluid as a function of $\tilde{n}$.

This leads to an energy-momentum tensor

$$T^{\mu\nu} = (\rho_0 + p_0 + \varphi^2)u^\mu u^\nu - \varphi^2 w^\mu w^\nu - (p_0 + \frac{1}{2}\varphi^2)g^{\mu\nu}, \tag{4.29}$$

where p_0 is the pressure of the perfect fluid component, which differs from the full string fluid pressure appearing as the coefficient of $g^{\mu\nu}$. The energy-momentum tensor for magnetohydrodynamics has been previously written in this form (e.g. [41]). We wish to emphasize that the variational principle for string fluids in terms of scalar fields can be applied to magnetohydrodynamics as well.

4.3 Pressureless string fluid

Besides the reduction to the perfect fluid, another simplification of the string fluid occurs when $\mathcal{L}$ only depends on φ and not n, a case previously studied by Kopczynski.[31] The inspiration behind the Kopczynski fluid came from the case where the pressure vanishes, in which case the string fluid further reduces to a model studied by Stachel in which the submanifolds behave as independent Nambu-Goto strings.[32] More recently it was shown that coarse-graining an interacting network of Nambu-Goto strings in the limit of local equilibrium leads to a pressureless string fluid where the submanifolds behave as wiggly strings[12].

To gain a better understanding of the connection between a pressureless fluid and classical strings, first note that $\tilde{n} = \tilde{F} \wedge dZ$ involves a factor of φ and so it may be helpful to define a factored number density ν,

$$n \equiv \varphi\nu \tag{4.30}$$

From (4.24), the condition for the pressure to vanish is

$$\begin{aligned}\mathcal{L} &= \varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_n + n\left(\frac{\partial\mathcal{L}}{\partial n}\right)_\varphi \\ &= \varphi\left(\frac{\partial\mathcal{L}}{\partial\varphi}\right)_\nu,\end{aligned}$$

which implies that the derivative $\mathcal{L}_{,\varphi}$ at constant ν is a function of ν alone, which we write as

$$\mathcal{L} \equiv -\varphi\, U(\nu). \tag{4.31}$$

Similarly we can define a modified tension of the same form as (4.25),

$$T \equiv U - U_{,\nu}\nu = \varphi^{-1}\tau, \tag{4.32}$$

so that the energy momentum tensor is just

$$T^{\mu\nu} = \varphi(Uu^\mu u^\nu - Tw^\mu w^\nu). \tag{4.33}$$

This notation is intentionally similar to that used by Carter in describing "barotropic" classical strings [37]. The difference is that Carter's formalism applies to a single string rather than a fluid foliated by worldsheets, and so any spacetime derivatives must be projected into the worldsheet directions. For instance, in Carter's formalism the condition for the simple bivector Σ to describe an integrable submanifold is given as,

$$h^\lambda{}_\mu \nabla_\lambda \Sigma^{\mu\nu} = 0. \tag{4.34}$$

This condition corresponds to our conservation of the F tensor

$$\nabla_\mu(\varphi\Sigma^{\mu\nu}) = 0. \tag{4.35}$$

In general it can be proven (using both (4.34) and (4.35)) that if there is a tensor $A^{\mu\cdots}$ where the index μ lies in the worldsheet, then the following statements are equivalent:

$$\begin{aligned}\nabla_\mu(\varphi A^{\mu\cdots}) &= 0 \\ h^\lambda{}_\mu \nabla_\lambda A^{\mu\cdots} &= 0.\end{aligned} \tag{4.36}$$

Since in the pressureless case the conservation of $T^{\mu\nu}$, $F^{\mu\nu}$, and n^μ are all of this form, we see that φ decouples from the equations of motion for the submanifolds themselves, and these latter equations only depend on derivatives along the worldsheets. So the motion of each individual submanifold may be solved for independently as a barotropic string described by the equation of state $U(\nu)$. Once Σ has been solved for, the string flux φ is determined by the initial values on any two-dimensional spacelike surface intersecting the submanifolds.

The connection between this variational approach and that of barotropic strings can be used to construct Lagrangians describing a theory with submanifolds acting as strings with an arbitrary equation of state. The simplest case would be trivial equation of state for Nambu-Goto strings where $U = \mu_0$, the constant string tension. The string fluid with submanifolds acting as Nambu-Goto strings is the Stachel model, and from (4.31) we see that the Lagrangian is just

$$\mathcal{L} = -\mu_0 \varphi = -\mu_0 \sqrt{(dX \wedge dY)^2}. \tag{4.37}$$

A slightly more complicated example would be a fluid description of a system of many Nambu-Goto strings [12] which has submanifolds behaving as wiggly strings satisfying the condition $UT = \mu_0^2$. By (4.32), this is described by the equation of state $U(\nu) = \mu_0 \sqrt{1+\nu^2}$. So the Lagrangian for that model is given by

$$\mathcal{L} = -\mu_0 \varphi \sqrt{1+\nu^2} = -\mu_0 \sqrt{\varphi^2 + n^2} = \tag{4.38}$$

$$= -\mu_0 \sqrt{\frac{1}{2}(dX \wedge dY)^2 - \frac{1}{3!}(dX \wedge dY \wedge dZ)^2}, \tag{4.39}$$

where again φ^2 and n^2 were rewritten in terms of the scalar fields X, Y, Z.

4.4 Variational principle

Until now the Lagrangian $\mathcal{L}$ has only been used to find the energy-momentum tensor. The conservation of $T^{\mu\nu}$ and the identities $d\tilde{n} = d\tilde{F} = 0$ are all that is needed for the equations of motion, but it is not clear that this is equivalent to requiring that the action S be invariant under variations of X, Y, Z. Writing the action explicitly in terms of these fields,

$$\begin{aligned} S &= \int dx^4 \sqrt{-g}\, \mathcal{L}\left(\varphi^2, n^2\right) \\ &= \int dx^4 \sqrt{-g}\, \mathcal{L}\left(\frac{1}{2}(dX \wedge dY)^2, -\frac{1}{3!}(dX \wedge dY \wedge dZ)^2\right). \end{aligned} \tag{4.40}$$

Since Z only appears in terms of its derivative, the field equation resulting from a variation δZ can be expressed as the conservation of a current Π_Z

$$\nabla_\nu (2\frac{\partial \mathcal{L}}{\partial n^2}\tilde{n}^{\lambda\mu\nu} X_{,\lambda} Y_{,\mu}) \equiv \nabla_\nu \Pi^\nu_Z = 0. \tag{4.41}$$

This can also be understood as the Noether current associated with translations in Z. Recalling the definition (4.10) of w and that of the chemical potential μ (4.2),

$$\Pi^\nu_Z = \varphi\mu w^\nu. \tag{4.42}$$

Due to the decoupling of φ through (4.36), for a pressureless fluid this is identical to the spacelike current that appears as a dual to n in Carter's work on classical strings (e.g. [38]). Here we see the connection to translation symmetry of Z, and see that an analogue also holds for string fluids with pressure.

The field equation corresponding to a variation δX can be written as

$$X_{,\kappa} Y_{,\mu} \nabla_\lambda \left(\frac{\partial \mathcal{L}}{\partial \varphi}\tilde{\Sigma}^{\lambda\mu}\right) - X_{,\kappa} Y_{[,\mu} Z_{,\nu]} \nabla_\lambda \left(\frac{\partial \mathcal{L}}{\partial n}\tilde{u}^{\lambda\mu\nu}\right) = 0. \tag{4.43}$$

Putting the δY and δZ equations in the same form and combining leads ultimately to the field equations

$$-\frac{3}{2}F^{\lambda\mu}\nabla_{[\kappa}\left(\frac{\partial \mathcal{L}}{\partial \varphi}\Sigma_{\lambda\mu]}\right) + 2n^\lambda \nabla_{[\kappa}\left(\frac{\partial \mathcal{L}}{\partial n}u_{\lambda]}\right) = 0. \tag{4.44}$$

For an ordinary particle fluid the first term vanishes and the second term is just the usual equations of motion (4.6). On the other hand the first term by itself appears also in Kopczynski's work.[31] These field equations can be shown to be equivalent to conservation of $T^{\mu\nu}$ by reversing the steps leading to (4.6). Although here we are considering a single current and a single string flux, adding additional dependences n_a and F_b in the Lagrangian simply leads to equations of the same form with a sum over the indices a, b.

Of course just as for Π_Z, the field equations for δX and δY can be understood as the conservation of the Noether currents Π_X, Π_Y associated with translations in X, Y. These are special cases of a larger group of symmetry transformations leaving the two-form $dX \wedge dY$ invariant. This group is equivalent to the symplectic transformations on the two-dimensional X, Y space. A symplectic transformation can be generated by an arbitrary function $H(X,Y)$, where for infinitessimal δt

$$\begin{aligned}\delta X &= +H_{,Y}\delta t\\ \delta Y &= -H_{,X}\delta t.\end{aligned} \tag{4.45}$$

Symmetry under these relabeling transformations corresponds to the conservation of the class of currents

$$\nabla_\mu(H_{,Y}\Pi^\mu_X - H_{,X}\Pi^\mu_Y) = 0, \tag{4.46}$$

which is in turn equivalent to certain conditions on Π_X, Π_Y,

$$\begin{aligned}\Pi^\mu_X Y_{,\mu} = \Pi^\mu_Y X_{,\mu} &= 0\\ \Pi^\mu_X X_{,\mu} &= \Pi^\mu_Y Y_{,\mu}.\end{aligned} \tag{4.47}$$

Similarly an arbitrary function of X, Y may be added to Z without changing the physical situation, and the conservation of the corresponding Noether currents is equivalent to the condition

$$\perp^\lambda{}_\mu\, \Pi^\mu_Z = 0. \tag{4.48}$$

where

$$\perp^\lambda{}_\mu = \delta^\lambda{}_\mu - u^\lambda u_\mu + w^\lambda w_\mu \tag{4.49}$$

is the orthogonal projection to 'worldsheets' spanned by u and w. And so any field theory with the same relabeling symmetries (which may depend on higher order derivatives of the scalar fields) will have field equations equivalent to the conservation of three currents Π_X, Π_Y, Π_Z satisfying the constraints above.

4.5 Domain Wall Fluid

The variational approach discussed in this paper can be easily generalized to different dimensions of spacetime and to different numbers of scalar fields. In such cases the submanifolds in the fluid may describe the world-volumes of higher dimensional branes instead of strings or particles. A simple case we will treat here is that of a single scalar field X in $3+1$ dimensional spacetime. In this case the $2+1$ dimensional submanifolds along which X is constant can describe the world-volume of two dimensional membranes or domain walls.

The gradient one-form $\tilde{G}_\mu \equiv X_{,\mu}$ annihilates the tangent vectors to the world-volume, so the orthogonal projector can be written as,

$$\perp_{\mu\nu} = -\frac{1}{\psi^2} X_{,\mu} X_{,\nu}, \tag{4.50}$$

where ψ is the magnitude of $X_{,\mu}$,

$$\psi^2 \equiv -\tilde{G}^\mu \tilde{G}_\mu. \tag{4.51}$$

This quantity ψ can be understood as the density of domain walls along their normal direction, and similarly to φ and n it may appear in the Lagrangian. As before, the divergence of the dual 3-form to $\tilde{G}$ vanishes

$$\begin{aligned} G^{\lambda\mu\nu} &\equiv \epsilon^{\lambda\mu\nu\rho} \tilde{G}_\rho \\ \nabla_\lambda G^{\lambda\mu\nu} &= 0, \end{aligned} \tag{4.52}$$

and separating G into its magnitude and direction

$$G^{\lambda\mu\nu} \equiv \psi \Sigma^{\lambda\mu\nu}, \tag{4.53}$$

the projector h onto the tangent space of the world-volume can be written

$$h^\mu{}_\nu = \frac{1}{2} \Sigma^{\mu\rho\sigma} \Sigma_{\nu\rho\sigma}. \tag{4.54}$$

Now considering the Lagrangian corresponding to the Stachel model (4.37)

$$\mathcal{L} = -\psi = -\sqrt{-g^{\mu\nu} X_{,\mu} X_{,\nu}}, \tag{4.55}$$

the energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{\psi} X_{,\mu} X_{,\nu} - \mathcal{L} g_{\mu\nu} \\ &= \psi(-\perp_{\mu\nu} + g_{\mu\nu}) = \psi h_{\mu\nu}. \end{aligned} \tag{4.56}$$

Expressing h in terms of Σ, the conservation of energy-momentum leads to

$$\Sigma^{\mu\rho\sigma} \nabla_\mu \Sigma_{\nu\rho\sigma} = 0, \tag{4.57}$$

This equation is the analogue of the equation $\Sigma^{\mu\rho}\nabla_\mu\Sigma_{\nu\rho} = 0$ appearing in the Stachel model of a string fluid [32]. And following exactly the same line of reasoning as in that paper we can choose three coordinates parametrizing the world-volume and define the maps ξ^μ embedding the world-volume in spacetime. Then Σ may be expressed in terms of ξ and ultimately we find

$$\xi^\mu_{,a}\nabla_\mu(\sqrt{-h}\xi^{;a}_\nu) = 0, \tag{4.58}$$

where h is now the determinant of the projector in the world-volume basis (i.e. it is the determinant of the pullback of the metric). This has exactly the same form as the Nambu-Goto equations of motion, except that a ranges over three coordinates on the world volume rather than two. And these are indeed the standard equations for a domain wall in the limit of zero thickness, see for instance [40].

As an aside, note that it is easy to also consider the Hamiltonian formulation of this theory of domain wall submanifolds. The conjugate momentum P to X is just the time component of the Noether current Π^μ_X associated to translations in X,

$$\begin{aligned} P &= \Pi^0_X \\ \Pi^\mu_X &= \frac{\partial \mathcal{L}}{\partial X_{,\mu}} = \frac{1}{\psi} g^{\mu\nu} X_{,\nu}, \end{aligned} \tag{4.59}$$

where, specializing to a Minkowski metric,

$$\psi^2 = \frac{|\nabla X|^2}{1+P^2}. \tag{4.60}$$

Then the Hamiltonian density is found to be

$$\begin{aligned} \mathcal{H} &= \dot{X}P + \psi \\ &= \psi P^2 + \psi \\ &= |\nabla X|\sqrt{1+P^2}, \end{aligned} \tag{4.61}$$

and Hamilton's equations just express the conservation of the current Π_X and its relation to the time derivative $X_{,0}$. This Hamiltonian may also be of interest in lower spacetime dimensions, where it describes a dust of Nambu-Goto strings or free particles.

Returning now to the Lagrangian formulation and a general metric, we can generalize the Lagrangian to an arbitrary function of ψ and that will lead to the appearance of non-vanishing pressure in the energy-momentum tensor just as in the string fluid case. So in the familiar theory of a massless scalar field $\mathcal{L} = g^{\mu\nu} X_{,\mu} X_{,\nu}$ (where $X_{,\mu}$ is spacelike) the surfaces along which X is constant act like domain walls under pressure.

Another direction of generalization is to reintroduce dependence of the Lagrangian on n in addition to ψ with both quantities expressed in terms of the same scalar field X. By construction the fluid velocity described by $\tilde{n}$ will be confined to the domain walls, just as before it was confined to the worldsheet submanifolds. In the same way, reintroducing dependence on φ in the Lagrangian will describe a string fluid confined to domain walls. This can be interpreted in a less obscure way as the Lagrangian for a perfect anisotropic fluid with a distinct pressure (or tension) in three characteristic spatial directions.

4.6 Clebsch Potentials

Some readers may be more familiar with the variational principle for perfect fluids in terms of Clebsch potentials. An irrotational velocity field can be described as the gradient of a scalar potential T. In discussing the vorticity of the fluid as in (4.6), it is appropriate to consider μu^λ rather than u alone, and so we take T to satisfy

$$\mu u_\lambda \equiv \mu_\lambda = \partial_\lambda T.$$

Then the fluid satisfies a variational principle with the Lagrangian equal to the pressure, which is taken to be a function of $\mu^2 = g^{\kappa\lambda}\partial_\kappa T \partial_\lambda T$. Note that this is formally similar to the domain wall fluid discussed in the previous section. The only difference is that $\partial_\lambda T$ is here taken to be timelike rather than spacelike.

If we wish to describe a fluid with nonvanishing vorticity we need to introduce additional scalar potentials. In a fluid with an entropy current in addition to a number density it is useful to consider four additional potentials as in a paper by Schutz [28]. We will delay the discussion of additional currents to the following section and consider a fluid with an equation of state depending on a single n. Then the vorticity is a simple bivector and we can describe the fluid with two additional scalar potentials X and Y,

$$\mu = dT + X dY. \tag{4.62}$$

As before, the pressure as a function of μ^2 can be taken as the Lagrangian, and variations of T, X and Y lead to the correct fluid equations [28].

The vorticity takes the form of the flux tensor $\tilde{F}$,

$$d\mu = dX \wedge dY \equiv \tilde{F}. \tag{4.63}$$

Previously we were taking $\tilde{F}$ to describe the flux carried by strings in some underlying network, and the vorticity is indeed the flux carried by vortex lines in a superfluid. The superfluid may be described on a large scale such that the individual vortex lines are coarse-grained and the vorticity is a continuous tensor [42][43]. One difference between this coarse-grained superfluid and an ordinary superfluid is that just as for a perfect string fluid, the thermodynamic quantities may depend on the magnitude of vorticity φ^2 as well as the quantity μ^2. Such a dependence appears already in the 'vortex fibration model' of Carter and Langlois [39]. In the remainder of this section we will show that their model of a superfluid at zero temperature follows from a simple modification of a perfect string fluid where the Lagrangian depends on μ and the scalar field T (instead of n and the earlier field Z).

First note that T does not respect the same symmetries as Z. There is still symmetry under shifts of T by a constant, which leads to a Noether current which will be identified as the ordinary fluid current n. But the quantity μ in (4.62) is not preserved if we add an arbitrary function of X, Y to T. Thus unlike the situation in (4.49), n is not in general orthogonal to $\tilde{F}$. In other words the vortex lines are not 'frozen into' the fluid flow, in contrast to field lines in ideal magnetohydrodynamics.

However there is a symmetry under a simultaneous change in X, Y and T. If X, Y are changed by a symplectic transformation (4.45), (4.62) will be preserved if T is changed by

$$\delta T = (H_X X - H)\delta t.$$

And besides μ and $\tilde{F}$, the quantity

$$\tilde{h}_{\lambda\mu\nu} \equiv (dX \wedge dY \wedge dT)_{\lambda\mu\nu} = (\mu \wedge \tilde{F})_{\lambda\mu\nu} = \mu\varphi\tilde{w}_{\lambda\mu\nu} \tag{4.64}$$

also satisfies this symmetry. So in general we may take the Lagrangian to also depend on

$$h^2 \equiv \frac{1}{3!}\tilde{h}^{\lambda\mu\nu}\tilde{h}_{\lambda\mu\nu}. \tag{4.65}$$

in addition to φ^2 and μ^2. The duplicate notation h is chosen in this context to agree with the notation for the helicity vector h in the Carter-Langlois model [39]. It is easy to see that a Lagrangian

$$\mathcal{L} = \mathcal{L}\left(\frac{1}{2}\left(dX \wedge dY\right)^2, (dT + XdY)^2, -\frac{1}{3!}\left(dX \wedge dY \wedge dT\right)^2\right). \tag{4.66}$$

leads to an energy-momentum tensor agreeing with that of Carter-Langlois. (Note that in Ref. [39] the Lagrangian was denoted by Ψ).

The purpose of this section is rather to demonstrate that variation of $\mathcal{L}$ by X, Y and T leads to an alternate variational principle to that of [39], which involves a Kalb-Ramond field rather than the scalar T and requires an extra term in the Lagrangian to enforce a constraint.

Defining the current n and the antisymmetric tensor λ through

$$\delta\mathcal{L} \equiv n^\rho \delta\mu_\rho + \frac{1}{2}\lambda^{\rho\sigma}\delta\tilde{F}_{\rho\sigma}, \tag{4.67}$$

it is clear that the equation of motion resulting from a variation δT is just the conservation

$$\nabla_\rho n^\rho = 0.$$

Variations by δX and δY respectively lead to

$$n^\rho Y_{,\rho} - \nabla_\rho(\lambda^{\rho\sigma}Y_{,\sigma}) = 0,$$
$$-\nabla_\rho(n^\rho X) - \nabla_\sigma(\lambda^{\rho\sigma}X_{,\rho}) = 0.$$

All these equations of motion for fields X, Y and T can be combined to obtain the equation of motion of the Carter-Langlois model

$$(n^\rho - \nabla_\sigma\lambda^{\sigma\rho})\tilde{F}_{\rho\tau} = 0, \tag{4.68}$$

which reduces the ordinary equation of motion for a perfect fluid (4.6), when the Lagrangian does not depend on $\tilde{F}$ (and thus $\lambda = 0$).

4.7 Additional Currents

In the standard treatment of perfect fluids the equation of state is often taken to depend on both the number density n and also a conserved entropy density n_s. The problem of how to extend the variational principle to fluids with multiple constituents needs to be addressed. The most obvious solution is to implement the current n_s by introducing an additional scalar field Z_s in the Lagrangian in the combination $\tilde{n}_s = dX \wedge dY \wedge dZ_s$. Then the two currents, n and n_s, may flow with different velocities, although both velocities will be confined to the same submanifolds.

If instead we wish for a current like entropy to flow with the same velocity u as n, there are two valid options. First consider a modification similar to that taken in the diffeomorphic approach to ordinary fluids [30]. The entropy per particle S is constant along the particle worldlines, so it is a function $S(X, Y, Z)$. The entropy density current is then

$$\tilde{n}_s = S(X, Y, Z)\tilde{n}, \tag{4.69}$$

which is conserved by construction and points in the direction u. A Lagrangian depending on n_s^2 then can be varied by X, Y, Z (but not S itself) as in (4.43). The extra dependence on n_s ultimately leads to an extra term in the equation of motion (4.44) of the form

$$2n_s^\lambda \nabla_{[\kappa} \left(\frac{\partial \mathcal{L}}{\partial n_s} u_{\lambda]} \right).$$

And this is just what is needed for conservation of the energy-momentum tensor (4.22) to hold if the equation of state also depends on n_s.

In this approach even for an arbitrary number of additional currents, we do not introduce any extra degrees of freedom in the theory in the sense of extra fields having conjugate momenta. However the function S, which physically depends on initial conditions, appears directly in the Lagrangian. This explicitly breaks the relabeling symmetries of X, Y, Z.

Previously the Noether current associated to shifts in Z was the dual current (4.42). It is indeed true that in the presence of additional currents (indexed by a) the dual current is not generalized to any gauge invariant Noether current. But a useful expression may still be derived from the δZ equation of motion,

$$n_a \nabla_\mu (\varphi \mu^a w^\mu) = 0. \tag{4.70}$$

This can also be derived without making use of the variational principle by using energy-momentum conservation $w_\mu \nabla_\nu T^{\mu\nu} = 0$, and the identity (4.36).

An alternate approach to implementing additional currents flowing with the velocity has been previously suggested [29]. We may introduce an extra scalar field θ, and allow the Lagrangian to also depend on θ through the combination

$$y \equiv \frac{1}{n}\epsilon^{\kappa\lambda\mu\nu}X_{,\kappa}Y_{,\lambda}Z_{,\mu}\theta_{,\nu}.$$

The equation of motion associated to $\delta\theta$ is

$$\nabla_\mu(\mathcal{L}_{,y}u^\mu) = 0, \tag{4.71}$$

and thus $\mathcal{L}_{,y}$ is interpreted as n_s. The quantity y itself is equal to the chemical potential associated to n_s, and the Lagrangian in this case is the Legendre transform of $-\rho$ in the n_s variable. In the specific case where n_s is interpreted as entropy, y is equal to the temperature T and $\mathcal{L}$ is the negative of the Helmholtz free energy. The field θ itself has appeared in the literature before as the quantity "thermasy" [28].

This approach introduces the additional degree of freedom θ, but maintains the relabeling symmetry in the Lagrangian. The Noether current associated to shifts in Z now contains a gauge dependent term involving θ

$$\Pi^\mu_Z = \varphi(\mu + ST)w^\mu - SF^{\mu\nu}\theta_{,\nu}. \tag{4.72}$$

However this dependence is eliminated upon taking the divergence

$$\begin{aligned}\nabla_\mu\left(SF^{\mu\nu}\theta_{,\nu}\right) &= F^{\mu\nu}\theta_{,\nu}\nabla_\mu S\\ &= \varphi T w^\mu \nabla_\mu S,\end{aligned}$$

where the second line makes use of the vanishing of the derivative of S in the u direction. So the conservation of the current Π_Z leads to the appropriate generalization (4.70) of the dual current conservation

$$\nabla_\mu \Pi^\mu_Z = \nabla_\mu(\varphi\mu w^\mu) + S\nabla_\mu(\varphi T w^\mu) = 0. \tag{4.73}$$

Finally note that both of these approaches to introducing extra currents can be easily generalized to introducing extra fluxes in the equation of state. For instance if we introduce dependence on the two fields θ^1, θ^2 in the combination

$$\upsilon \equiv \frac{1}{\varphi}\epsilon^{\kappa\lambda\mu\nu}X_{,\kappa}Y_{,\lambda}\theta^1_{,\mu}\theta^2_{,\nu},$$

the two Noether currents associated with the new fields are equivalent to the conservation of a single antisymmetric tensor $F_s = \mathcal{L}_{,v}\Sigma$. The conservation of F_s and F implies that $\mathcal{L}_{,v}/\varphi$ is constant on the worldsheets, and so we can represent it by a function $S(X,Y)$. Alternately we could introduce $S(X,Y)$ directly in the Lagrangian through a dependence on $\varphi_s^2 = S^2\varphi^2$. And again the Lagrangians for the two distinct approaches to introducing fluxes are simply related through Legendre transforms.

Chapter 5

Dissipative fluids of strings

5.1 Introduction

The following chapter is taken from the paper 'Dissipative String Fluids.' [14]

Networks of one-dimensional strings appear in a variety of contexts. In particular, networks of quantized vortex lines appear in turbulent quantum fluids, and networks of cosmic strings may have formed in a symmetry breaking phase transition in the early universe. These networks have been extensively studied using numerical models which track the motion of individual strings in the network, as in for instance the vortex-filament model of Schwartz [2] or the Smith-Vilenkin model for cosmic strings [3]. For many purposes it may be useful to instead consider a 'macroscopic' perspective in which individual strings are coarse-grained in a fluid approximation. In the context of quantum turbulence, such an approximation underlies the Hall-Vinen-Bekharevich-Khalatnikhov equations [5] which describe the net vorticity of the network as a continuous field interacting with the usual two-fluid model of a superfluid. On the other hand, in the cosmic string context the dynamics of the strings themselves are often considered independently from any interaction with external fields. Coarse-graining such a network leads to an independent 'string fluid' which may exhibit interesting properties distinct from any additional interactions with other fluids.

The individual strings in the network carry a conserved flux. For instance the vortex lines in a superfluid carry quantized angular momentum and the topological defects in the Abelian-Higgs model carry magnetic flux. In the coarse-grained fluid the conservation of flux is manifested as the conservation of an antisymmetric tensor F:

$$\nabla_\mu F^{\mu\nu} = 0. \tag{5.1}$$

In a fluid of strings carrying magnetic flux, F is just the dual of the electromagnetic field tensor, and the vanishing of its divergence is just a statement of the homogeneous Maxwell equations. But in fact for any fluid of directed strings there is a conservation law for a tensor F which describes the topological flux of the strings [11]. It is tempting at this point to point out the similarity to magnetohydrodynamics (MHD) which is another example of a fluid with a conserved magnetic flux. A connection between Nambu-Goto strings and MHD has in fact been previously noticed by Olesen [25]. In Sec. 5.2.2 we will show through quite different methods that ideal MHD is a particular case of what we call a 'perfect string fluid'. Formally, a coarse-grained network of strings has many similarities with a plasma, but there are differences in the equation of state of the fluid at equilibrium.

Some readers may here question the idea of an equilibrium for cosmic string networks at all. Through reconnection events the small-scale structure on long strings tends to lead to the production of small loops. It was realized early on from numerical simulations that the reverse process whereby small loops attach to long strings is much less effective for densities below a critical density.[3][4] Given a minimum energy cutoff beyond which small loops are restricted from fragmenting, most of the energy will flow into loops of energy comparable to the cutoff size. So any equilibrium properties will be cutoff dependent, and thus artificial in a sense. Of course the idea of separating the string dynamics from all other interactions is artificial as well, and loops near the cutoff may leave the system through various decay channels.

But what the same numerical simulations do show is that very different initial conditions will lead to the same cutoff-dependent equilibrium state, which depends on the energy density as well any net flux of the strings through the system space. And the statistics of the equilibrium states in the numerical simulations agree with analytical calculations by Mitchel and Turok [6] which involve notions of temperature and entropy for the string networks. The temperature of the equilibrium states remains near the Hagedorn temperature for a very wide range of densities [7]. This may suggest that the decay of small loops and wiggles can be accounted for as the flow of heat from a hot string fluid out of thermal equilibrium with the environment.

In any case, in this paper we will restrict our investigation to the dynamics of an isolated string fluid, and take a macroscopic perspective in which an equation of state is given without reference to an underlying string network. Indeed the example of magnetohydrodynamics shows that what we here call a string fluid may have nothing to do with strings at all on a more microscopic level. The requirements of thermodynamics are then shown to lead to dissipative terms in the fluid equations which correspond to the formation of small-scale structure in an underlying string network.

The paper is organized as follows. In Sec. 5.2 the concept of a perfect string fluid is reviewed. A full treatment emphasizing the variational principle satisfied is found in [13], and the concept has also been studied in the context of blackfolds [26]. The dissipative equations will depend on the equation of state in equilibrium, so two particular cases of a perfect string fluid are discussed. In Sec. 5.2.1 an idealized equation of state for a network of Nambu-Goto strings is reviewed. In Sec. 5.2.2 it is shown that ideal MHD is another example of a perfect string fluid.

Section 5.3.1 begins the discussion of dissipative effects by discussing the ambiguities in choosing the flow velocity and field line direction for a general fluid. Given such a choice, the conserved tensors are broken up into equilibrium and dissipative parts. In Sec. 5.3.2 the entropy current is determined, and the positivity of entropy production is used to find the explicit form of the dissipative terms.

The dissipative parts of the energy-momentum tensor are much the same as for an ordinary fluid, but the dissipative parts of the conserved flux tensor are discussed in 5.3.3. Entropy production due to the curvature of the field lines is discussed in terms of plausible effects in an underlying network of cosmic strings. The nonrelativistic limit of the theory is taken and compared to ordinary resistive magnetohydrodynamics. The dissipative correction to the electric field can be seen as resulting from Ohm's law, but there is an additional term coupling the electric field to temperature gradients.

In Sec. 5.3.4 necessary conditions for the fluid to be at equilibrium are derived. As for ordinary fluids, there is a timelike Killing vector proportional to the velocity. In the string fluid there is also an irrotational vector field proportional to the field line direction. In Sec. 5.3.5 an extension to a higher order dissipative theory similar to the Israel-Stewart model [8] is discussed. The equation describing heat flow along a string is corrected to be hyperbolic, and the speed of second sound is calculated for the idealized cosmic string model discussed in Sec. 5.2.1.

5.2 Perfect String Fluids

An ordinary perfect fluid involves one or more conserved currents n_a^μ (indexed by a) which represent extensive quantities such as electric charge, particle number, or entropy. The currents flow in the direction of the timelike velocity u of the fluid,

$$n_a^\mu = n_a u^\mu, \tag{5.2}$$

and we will here use a $(+,-,-,-)$ signature.

The thermodynamics of the fluid is specified by giving the energy density ρ as a function of the magnitudes n_a. Then the chemical potentials m^a are defined as

$$m^a \equiv \frac{\partial \rho}{\partial n_a}, \tag{5.3}$$

and the pressure p is defined essentially as a Legendre transform,

$$\rho = -p + m^a n_a \tag{5.4}$$

$$dp = n_a dm^a \tag{5.5}$$

Given these quantities, the energy-momentum tensor is just

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}, \tag{5.6}$$

and the fluid equations are equivalent to the conservation laws

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu n_a^\mu = 0. \tag{5.7}$$

Note that if one pair of density and chemical potential is singled out as the entropy density s and temperature T, the remaining conservation laws (5.7) and the expression for the derivatives of the pressure (5.5) can be used to prove the conservation of s,

$$\begin{aligned} u_\mu \nabla_\nu T^{\mu\nu} &= \nabla_\nu (m^a n_a + Ts) u^\nu - u^\nu \nabla_\nu p \\ &= T \nabla_\nu s u^\nu. \end{aligned} \tag{5.8}$$

Similar expressions will be useful in extending to the dissipative case.

A string fluid also involves the conservation of at least one antisymmetric flux tensor F,

$$\nabla_\mu F^{\mu\nu} = 0. \tag{5.9}$$

In the case of a perfect string fluid, F is a simple bivector that can be written as the alternating product of two vectors. Further, the fluid velocity u is in the linear space spanned by these vectors. The velocity u can be used to define a normalized spacelike direction w and a positive magnitude φ,

$$\varphi\, w^\mu \equiv F^{\mu\nu} u_\nu \tag{5.10}$$

$$u^\mu u_\mu = -w^\mu w_\mu = 1 \tag{5.11}$$

$$u^\mu w_\mu = 0. \tag{5.12}$$

Together, u and w determine the directional part Σ of F,

$$\Sigma^{\mu\nu} \equiv w^\mu u^\nu - u^\mu w^\nu \tag{5.13}$$

$$F^{\mu\nu} = \varphi \Sigma^{\mu\nu}. \tag{5.14}$$

It will also be useful to define the projector h onto the space spanned by u and w, and its orthogonal complement $\perp$,

$$\begin{aligned} h^\mu{}_\nu &= \Sigma^{\mu\rho}\Sigma_{\rho\nu} \\ &= u^\mu u_\nu - w^\mu w_\nu \end{aligned} \tag{5.15}$$

$$\begin{aligned} \perp^\mu{}_\nu &= \tilde{\Sigma}^{\mu\rho}\tilde{\Sigma}_{\nu\rho} \\ &= \delta^\mu{}_\nu - u^\mu u_\nu + w^\mu w_\nu, \end{aligned} \tag{5.16}$$

where we are using tildes to denote the Hodge dual,

$$\tilde{\Sigma}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma}. \tag{5.17}$$

The dual $\tilde{F}$ of F itself is a two-form that can be integrated to give the net flux carried by the strings across a surface. The magnitude φ is thus a measure of this flux and it is taken to be a thermodynamic variable on the same footing as the densities n_a. The conjugate chemical potential to φ is denoted by μ,

$$\mu \equiv \frac{\partial \rho}{\partial \varphi}. \tag{5.18}$$

And the pressure for a string fluid now involves $\mu\varphi$,

$$\rho = -p + m^a n_a + \mu\varphi. \tag{5.19}$$

In an earlier paper [13] it was shown that a quite general variational principle leads to an energy-momentum tensor of the form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - (\tau + p)w^\mu w^\nu - pg^{\mu\nu}, \tag{5.20}$$

where τ is a thermodynamic potential related to the tension of the strings,

$$\tau \equiv -p + \mu\varphi. \tag{5.21}$$

The equations of motion of the perfect string fluid are then equivalent to the conservation of $T^{\mu\nu}$ and all currents and fluxes (5.7)(5.1).

5.2.1 Wiggly string fluid

We will now review some particular examples of string fluids. Directly coarse-graining a network of Nambu-Goto strings leads to T and F tensors expressed in terms of correlations between the (non-unit vector) string velocity U and the tangent vector to the string W.[11][12]

$$\begin{aligned} T^{\mu\nu} &= \langle U^\mu U^\nu - W^\mu W^\nu \rangle \\ F^{\mu\nu} &= \langle W^\mu U^\nu - U^\mu W^\nu \rangle. \end{aligned} \tag{5.22}$$

The vectors U and W are properties of the individual strings in the network and the brackets denote an integration over a coarse-graining volume. There are sixteen independent components of these tensors, and so the conservation of the T and F tensors alone does not fully specify the system.

The extra assumption needed was suggested by Vanchurin's kinetic theory of a gas of string segments. [9] This model suggested that the strings would equilibriate to a state in which there are no correlations between the statistics of right and left movers. The principle that the string fluid should everywhere locally be in an equilibrium of this form allowed for the correlations in (5.22) to be factored into the average string velocity field $\bar{U}$ and the average tangent vector field $\bar{W}$.

At this point we will note that for a general string fluid the conservation $\nabla_\mu F^{\mu\nu} = 0$ together with the condition that F be a simple bivector implies that spacetime can be foliated by two-dimensional manifolds that are everywhere tangent to the linear subspace defined by the projector h in (5.15).[12] Since $\bar{U}$ and $\bar{W}$ lie in this tangent space, it is tempting to interpret the manifolds as the worldsheets of 'macroscopic strings' which point in the direction of the field lines of $\bar{W}$ and propagate with velocity $\bar{U}$.

Ultimately these fields can be expressed in terms of the variables φ, u, w in the present paper, and the energy-momentum tensor takes the form

$$T^{\mu\nu} = \varphi(Mu^\mu u^\nu - Tw^\mu w^\nu), \tag{5.23}$$

where the quantities M and T can be respectively interpreted as the mass-per-length and tension of the macroscopic strings.

In fact M and T have exactly the same form as the mass-per-length and tension of a single 'wiggly string' which can be described as an ordinary Nambu-Goto string with small-scale perturbations integrated out.[15][16] In the string fluid, the wiggles of the macroscopic strings may also involve disconnected loops smaller than the coarse-graining scale. The coarse-grained wiggles appear in the string fluid as a conserved 'wiggle number density' n, in terms of which the equation of state can be expressed as

$$\rho(n, \varphi) = \varphi M = \sqrt{(\mu_0 \varphi)^2 + n^2}, \tag{5.24}$$

where μ_0 is the mass per length of a Nambu-Goto string.[13]

Given that n describes structure below the macroscopic scale, and that the tendency towards production of small loops should monotonically increase n, this strongly suggests that n is proportional to the entropy density s:

$$\rho = \sqrt{(\mu_0 \varphi)^2 + (T_H s)^2}, \tag{5.25}$$

where T_H is some constant of proportionality. In the limit as s goes to infinity, the temperature T goes to the finite value T_H,

$$T \equiv \left(\frac{\partial \rho}{\partial s}\right)_\varphi \to T_H, \tag{5.26}$$

which suggests that we identify T_H as the Hagedorn temperature. For a single wiggly string there is also a corresponding conserved current and equation of state (differing by a factor of φ), and the identification of this current as the entropy has been previously made [17]. Even so the entropy is conserved in both the dynamics of wiggly strings and in perfect string fluids. The idea will be extended in this paper by introducing dissipative effects leading to increases in entropy density.

5.2.2 Magnetohydrodynamics

A relativistic formulation of magnetohydrodynamics is given for instance by Harris [19].. The energy-momentum tensor is simply the sum of a fluid part and an electromagnetic part,

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\mathrm{m}} + T^{\mu\nu}_{\mathrm{EM}} \\ &= (\rho + p) u^\mu u^\nu - p g^{\mu\nu} - \tilde{F}^{\mu\rho} \tilde{F}^{\nu}{}_{\rho} + \frac{1}{4} g^{\mu\nu} \tilde{F}^{\rho\sigma} \tilde{F}_{\rho\sigma}. \end{aligned} \tag{5.27}$$

Taking the divergence,

$$u_\nu \nabla_\mu T^{\mu\nu} = T \nabla_\mu s u^\mu - u_\nu \tilde{F}^{\nu}{}_{\rho} j^\rho = 0, \tag{5.28}$$

where we have used the homogenous Maxwell equations (5.1) and the expression for divergence of entropy (5.8), and the current j is defined by the Maxwell equations,

$$j^\mu \equiv \nabla_\mu \tilde{F}^{\mu\rho}. \tag{5.29}$$

The positivity of entropy production,

$$\nabla_\mu s u^\mu \geq 0, \tag{5.30}$$

will be satisfied if in fact the current is given by

$$j^\rho = q u^\rho + \sigma \tilde{F}^{\mu\rho} u_\mu, \tag{5.31}$$

where σ is a positive scalar and q can be arbitrary. But in the rest frame of the fluid $\tilde{F}^{\mu\nu}u_\mu$ is just the electric field, so this is just a statement of Ohm's law [18]. We will later return to this point, but presently we will consider the isentropic case of ideal magnetohydrodynamics.

For entropy to be conserved in (5.28) the electric field must vanish in the rest frame,

$$\tilde{F}^{\mu\nu}u_\nu = 0. \tag{5.32}$$

This is just the well-known condition for frozen-in magnetic field lines, but for our purposes it implies that $\tilde{F}$ and its dual F are simple bivectors, and that u is in the linear subspace spanned by F. So we can define φ and w as before, noting that they can be interpreted as the magnitude and direction of the magnetic field in the rest frame.

The energy-momentum tensor can be simplified using the expression for the orthogonal projector (5.16),

$$\begin{aligned} T^{\mu\nu} &= (\rho + p)u^\mu u^\nu - pg^{\mu\nu} - \varphi^2 \perp^{\mu\nu} + \frac{1}{2}g^{\mu\nu}\varphi^2 \\ &= (\rho + p + \varphi^2)u^\mu u^\nu - \varphi^2 w^\mu w^\nu - (p + \frac{1}{2}\varphi^2)g^{\mu\nu}. \end{aligned} \tag{5.33}$$

So if the total equation of state is taken as

$$\rho_{\text{total}} = \rho + \frac{1}{2}\varphi^2, \tag{5.34}$$

then the other thermodynamic quantities are found to be

$$\mu = \varphi \tag{5.35}$$

$$p_{\text{total}} = p + \frac{1}{2}\varphi^2 \tag{5.36}$$

$$\tau + p_{\text{total}} = \varphi^2, \tag{5.37}$$

showing that this is indeed an example of a perfect string fluid.

Note that the form of the energy density is just what we would expect from the variational principle for perfect string fluids [13]. There it was shown that the total energy density ends up being the negative of the Lagrangian. And the extra term in the energy density is just the negative of the usual Lagrangian for electromagnetism

$$-\frac{1}{4}\tilde{F}^{\rho\sigma}\tilde{F}_{\rho\sigma} = -\frac{1}{2}\varphi^2.$$

5.3 Dissipative String Fluids

5.3.1 Tensor decomposition

In a more general string fluid the conservation equations for T, F, and any additional conserved currents n_a still hold, but the tensors are no longer in the equilibrium forms (5.14)(5.20). Just as for an ordinary dissipative fluid, there is no longer a single preferred fluid velocity u. We may take the fluid velocity to be in the direction of the timelike eigenvector of energy-momentum tensor (a choice known as the 'Landau-Lifshitz frame' [20]) or we may choose the velocity to be in the direction of one of the currents (known as the 'Eckart frame' [18]) —the directions no longer coincide in general. In a string fluid we are now faced with the additional problem that the tensor F may no longer be a simple bivector, and so there is ambiguity in how to define w.

We may still select u and w as orthonormal vectors in the two-dimensional timelike eigenspace of $F^{\mu\rho}F_{\rho\nu}$. In general the fluid velocity from the Eckart or Landau-Lifshitz frames will not lie in this space so this can define a distinct third possible choice for velocity. As we will see, this frame will have some similarities to the Eckart frame. To distinguish the two cases, the ordinary Eckart frame will be referred to as the 'particle frame' and the choice of velocity from this eigenspace as the 'string frame'.

There is also the difficulty that none of the frames above satisfy the integrability conditions of the perfect string fluid. We can no longer foliate spacetime by worldsheets everywhere tangent to u and w. However the conservation of F does imply that we can define a gauge potential A,

$$\tilde{F} \equiv dA.$$

And by Darboux's theorem A can be written in terms of four scalar fields X_1, Y_1, X_2, Y_2,

$$A \equiv X_1\, dY_1 + X_2\, dY_2.$$

So then $\tilde{F}$ can be decomposed into two simple two-forms with vanishing exterior derivatives

$$\tilde{F} = dX_1 \wedge dY_1 + dX_2 \wedge dY_2.$$

These two-forms each annihlate a two-dimensional space which does satisfy the integrability condition. So this could be used to define yet another natural choice of u and w which preserves the integrability condition.

We restrict our attention to fluids that are sufficiently close to equilibrium so that the difference between these frames is 'small'. We will be more precise on this point later, where frame invariance will be used to restrict higher order dissipative terms in the theory. For now, given a choice of u and w, we can define ρ, φ, and n_a from the nonequilibrium tensors,

$$\begin{aligned} \rho &\equiv T^{\mu\nu} u_\mu u_\nu \\ \varphi &\equiv F^{\mu\nu} u_\mu w_\nu \\ n_a &\equiv n_a^\mu u_\mu . \end{aligned} \tag{5.38}$$

These values can be used to define the other thermodynamic quantities through the equilibrium equation of state. And so T and F can be decomposed into an equilibrium tensor and a nonequilibrium correction. The nonequilibrium correction may further be decomposed into parts parallel and orthogonal to u and w.

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu - \mu\varphi w^\mu w^\nu - p g^{\mu\nu} + 2q^{(\mu} u^{\nu)} + \pi^{\mu\nu} \tag{5.39}$$

$$F^{\mu\nu} = \varphi \Sigma^{\mu\nu} - 2u^{[\mu}\lambda^{\nu]} + 2w^{[\mu}\nu^{\nu]} + G^{\mu\nu} \tag{5.40}$$

$$n_a^\mu = n_a u^\mu + N_a w^\mu + \nu_a^\mu , \tag{5.41}$$

and q and π are further split,

$$q^\mu \equiv Q_L w^\mu + q_T^\mu \tag{5.42}$$

$$\pi^{\mu\nu} \equiv -\Pi_L w^\mu w^\nu + \Pi_T \perp^{\mu\nu} - 2w^{(\mu}\pi_L^{\nu)} + \pi_T^{\mu\nu} . \tag{5.43}$$

The vectors and tensors $\lambda, \nu, G, q_T, \pi_L, \pi_T, \nu_a$ are all fully orthogonal to u and w, and π_T is defined to be traceless. It should be emphasized that this is simply a decomposition of the tensors, and there is no loss of generality at this point.

If u and w are taken from our preferred frames some of these pieces vanish. In the string Eckart frame, u and w are chosen from an eigenspace so that both vectors λ and ν in F vanish. There is still some freedom in our choice of u, but there is a unique u such that the longitudinal heat flow Q_L vanishes.

In the Landau-Lifshitz frame the vector ν is nonzero but all heat flow components q vanish. Specifying w through

$$\varphi w^\mu \equiv F^{\mu\nu} u_\nu, \tag{5.44}$$

the vector λ vanishes as well.

5.3.2 Entropy current

The entropy density s is defined through the equilibrium equation of state, and satisfies the usual thermodynamic identities

$$\begin{aligned} s &= \frac{p}{T} + \frac{1}{T}\rho - \frac{\mu}{T}\varphi - \frac{m^a}{T} n_a \\ ds &= \frac{1}{T} d\rho - \frac{\mu}{T} d\varphi - \frac{m^a}{T} dn_a. \end{aligned} \tag{5.45}$$

It will be useful to promote the derivatives of the entropy to vectors,

$$\beta^\mu \equiv \frac{1}{T} u^\mu \tag{5.46}$$

$$\alpha^\mu \equiv \frac{\mu}{T} w^\mu. \tag{5.47}$$

Then the equilibrium entropy current can be written in terms of the equilibrium tensors T_0, F_0, n_{a0}

$$s u^\mu = p\beta^\mu + \beta_\nu T_0^{\mu\nu} - \alpha_\nu F_0^{\mu\nu} - \frac{m^a}{T} n_{a0}^\mu \tag{5.48}$$

$$d(s u^\mu) = \beta_\mu dT_0^{\mu\nu} - \alpha_\nu dF_0^{\mu\nu} - \frac{m^a}{T} dn_{a0}^\mu. \tag{5.49}$$

Closely following the approach of Israel and Stewart [8] we then make the assumption that the derivatives of nonequilibrium entropy current s^μ satisfy the same relation with the nonequilibrium tensors,

$$ds^\mu = \beta_\mu dT^{\mu\nu} - \alpha_\nu dF^{\mu\nu} - \frac{m^a}{T} dn_a^\mu. \tag{5.50}$$

The entropy current is taken to be a function of the components of T, F, n_a, and we can expand about the equilibrium point T_0, F_0, n_{a0}. To first order,

$$\begin{aligned} s^\mu &= s u^\mu + \beta_\mu (T - T_0)^{\mu\nu} - \alpha_\nu (F - F_0)^{\mu\nu} - \frac{m^a}{T} (n_a - n_{a0})^\mu \\ &= s u^\mu + \frac{1}{T} q^\mu - \frac{\mu}{T} \nu^\mu - \frac{m^a}{T} \nu_a^\mu. \end{aligned} \tag{5.51}$$

Comparison with (5.45) suggests q is naturally interpreted as a heat vector describing the transport of energy in the rest frame. The currents ν and ν_a respectively describe the transport of flux and charge in the rest frame through diffusion.

Expressions for the dissipative quantities appearing in the theory can now be determined by requiring that the entropy production be non-negative

$$\nabla_\mu s^\mu \geq 0. \tag{5.52}$$

The divergence of s can be found through similar manipulations as those leading to the conservation of entropy in the perfect fluid (5.8). For brevity at this point we will consider a theory with no dependence on conserved currents n_a, and choose the Landau-Lifshitz frame so that the heat vector q vanishes. These aspects of the derivation are no different than that for particle fluids (see e.g. [20]) and can be easily derived for a string fluid in the same way. Beginning with the dissipative energy-momentum tensor (5.39):

$$\begin{aligned} u_\nu \nabla_\mu T^{\mu\nu} &= \nabla_\mu(\rho + p)u^\mu + \mu\varphi w^\mu w^\nu \nabla_\mu u_\nu - u^\mu \nabla_\mu p - \pi^{\mu\nu}\nabla_\mu u_\nu \\ &= T\nabla_\mu s u^\mu + \mu \nabla_\mu \varphi u^\mu - \mu\varphi h^{\mu\nu}\nabla_\mu u_\nu - \pi^{\mu\nu}\nabla_\mu u_\nu \end{aligned} \tag{5.53}$$

where h is the projection operator defined in (5.15). If it were still true that $F = \varphi\Sigma$ the middle terms involving φ would cancel using a relation derived in [13]. This would be one way to show entropy is conserved in a perfect string fluid. But now the relation is modified due to dissipative terms in F,

$$\begin{aligned} \nabla_\mu \varphi u^\mu &= \nabla_\mu(\varphi \Sigma^{\mu\lambda} w_\lambda) \\ &= \nabla_\mu(F^{\mu\lambda} w_\lambda - \nu^\mu) \\ &= F^{\mu\lambda}\nabla_\mu w_\lambda - \nabla_\mu \nu^\mu \\ &= \varphi\Sigma^{\mu\lambda}\nabla_\mu w_\lambda + (2w^{[\mu}\nu^{\lambda]} + G^{\mu\lambda})\nabla_\mu w_\lambda - \nabla_\mu \nu^\mu \\ &= \varphi h^\kappa{}_\mu \nabla_\kappa \Sigma^{\mu\lambda} w_\lambda - \varphi w_\lambda h^\kappa{}_\mu \nabla_\kappa \Sigma^{\mu\lambda} + \dots . \end{aligned}$$

It can be shown (for instance by explicitly writing Σ and h in terms of u and w) that $h^\kappa{}_\mu \nabla_\kappa \Sigma^{\mu\lambda}$ is orthogonal to w. So the second term above vanishes, and returning to the derivation (5.53),

$$
\begin{aligned}
0 &= \nabla_\mu s u^\mu + \frac{\mu}{T}[(2w^{[\mu}\nu^{\lambda]} + G^{\mu\lambda})\nabla_\mu w_\lambda - \nabla_\mu \nu^\mu] - \frac{1}{T}\pi^{\mu\nu}\nabla_\mu u_\nu \\
&= \nabla_\mu (s u^\mu - \frac{\mu}{T}\nu^\mu) + \frac{\mu}{T}(2w^{[\mu}\nu^{\lambda]} + G^{\mu\lambda})\nabla_\mu w_\lambda + \nu^\mu \nabla_\mu \frac{\mu}{T} - \frac{1}{T}\pi^{\mu\nu}\nabla_\mu u_\nu \\
&= \nabla_\mu s^\mu + \frac{\mu}{T} G^{\mu\lambda}\nabla_\mu w_\lambda + \nu^\mu(\nabla_\mu \frac{\mu}{T} + \frac{\mu}{T} w^\lambda \nabla_\lambda w_\mu) - \frac{1}{T}\pi^{\mu\nu}\nabla_\mu u_\nu . \qquad (5.54)
\end{aligned}
$$

Now the second law (5.52) will be satisfied if each of the other terms is strictly negative. So we choose ν and G to have the form

$$
\nu^\mu = \xi_T \perp^{\mu\rho} (\nabla_\rho \frac{\mu}{T} + \frac{\mu}{T} w^\sigma \nabla_\sigma w_\rho) \qquad (5.55)
$$

$$
G^{\mu\nu} = -\xi_L \frac{\mu}{T} \perp^{\mu\rho} \perp^{\nu\sigma} \nabla_{[\rho} w_{\sigma]}, \qquad (5.56)
$$

where the coefficients ξ_T, ξ_L are positive scalars. Breaking up the viscous tensor π into its parts as in (5.43) we find a series of terms each of which is set to be negative by choosing

$$
\Pi_L = -3\,\zeta_L\, w^\rho w^\sigma \nabla_\rho u_\sigma \qquad (5.57)
$$

$$
\Pi_T = \frac{3}{2}\,\zeta_T \perp^{\rho\sigma} \nabla_\rho u_\sigma \qquad (5.58)
$$

$$
\pi_L^\mu = 2\,\eta_L \perp^{\mu\rho} w^\sigma \nabla_{(\rho} u_{\sigma)} \qquad (5.59)
$$

$$
\pi_T^{\mu\nu} = 2\,\eta_T \left(\perp^{\mu\rho}\perp^{\nu\sigma} - \frac{1}{2} \perp^{\mu\nu}\perp^{\rho\sigma} \right) \nabla_{(\rho} u_{\sigma)}, \qquad (5.60)
$$

with positive coefficients $\zeta_L, \zeta_T, \eta_L, \eta_T$. In principle the physics in the longitudinal direction w may be different from the transverse directions, which is why there are twice as many dissipative coefficients as for an isotropic fluid. The normalization of the coefficients is chosen so that if the physics were isotropic $\zeta_L = \zeta_T$ would be the usual bulk viscosity coefficient and $\eta_L = \eta_T$ the usual shear viscosity coefficient.

Note that the longitudinal viscosity vector π_L (5.59) potentially represents two distinct physical effects. One is due to changes in the transverse velocity along a single macroscopic string or field line. The other effect is due to differences in the longitudinal velocities of nearby strings. Due to the symmetry of the energy-momentum tensor these must be described by the same viscosity coefficient, but if we allow for intrinsic angular momentum these could in principle be different.

For completeness we may also consider a frame in which the heat vector q does not vanish. Following the same line of derivation there would be an extra entropy production term

$$-q^\mu(\nabla_\mu \frac{1}{T} + \frac{1}{T} u^\nu \nabla_\nu u_\mu)$$

in (5.54). The two pieces of q are thus set as

$$Q_L = \kappa_L w^\mu (\nabla_\mu T - T u^\nu \nabla_\nu u_\mu) \tag{5.61}$$

$$q_T^\nu = \kappa_T \perp^{\mu\nu} (\nabla_\mu T - T u^\nu \nabla_\nu u_\mu), \tag{5.62}$$

where κ_L, κ_T are the positive heat conductivity coefficients. The apparent difference in sign from the Fourier heat conduction law is just due to the signature of the metric.

5.3.3 Dissipation in F

Besides the appearance of an anisotropic direction, the dissipative terms in T are essentially the same as for an ordinary fluid. What may require some interpretation are the dissipative terms (5.55)(5.56) in F,

$$F^{\mu\nu} = 2w^{[\mu}(\varphi u + \nu)^{\nu]} + G^{\mu\nu}. \tag{5.63}$$

The tensor is here written in a form emphasizing the analogy to ordinary particle currents (5.41). The velocity u_E in the string Eckart frame where ν does not appear explicitly in F is clearly given by

$$u_E \approx u + \frac{1}{\varphi}\nu, \tag{5.64}$$

where this is only an equality to first order in the dissipative fields. Following a similar line of reasoning to Landau-Lifshitz [20], we replace the velocity in the first term of the energy-momentum tensor,

$$(\rho + p)u^\mu u^\nu \approx (\rho + p)u_E^\mu u_E^\nu - 2\frac{\rho + p}{\varphi}\nu^{(\mu} u_E^{\nu)}. \tag{5.65}$$

So the heat vector in the Eckart frame is approximately

$$q_E = -\frac{\rho + p}{\varphi}\nu. \tag{5.66}$$

Substituting the expression (5.55) for ν and ignoring the term due to curvature of w,

$$q_E = \frac{\rho + p}{\varphi}\xi_T \nabla_\perp \frac{\mu}{T}. \tag{5.67}$$

So by the thermodynamic identity

$$Td(\frac{\mu}{T}) = -\left(\frac{\rho+p}{\varphi T}\right) dT + dp, \tag{5.68}$$

we can make the identification

$$\xi_T = \left(\frac{\varphi T}{\rho + p}\right)^2 \kappa_T. \tag{5.69}$$

So ξ_T can be related to heat conductivity —but this is not the only way to understand ν, and the interpretation of G is still obscure. This may be clarified by taking the nonrelativistic limit:

$$\begin{aligned} \nabla_\mu &= (c^{-1}\partial_t, \nabla_i) \\ u^\mu &\to (1, c^{-1}\mathbf{v}) \qquad (5.70) \\ w^\mu &\to (c^{-1}\mathbf{v}\cdot\mathbf{w}, \mathbf{w}), \qquad (5.71) \end{aligned}$$

where $\mathbf{w}$ is a unit vector. The metric is taken to be the Minkowski metric, so as c goes to infinity the time components of $\perp^{\mu\nu}$ go to zero. Thus the time components of ν and G vanish, and we will take the spatial components to be of order c^{-1}. So in the nonrelativistic limit $\nabla_\mu F^{\mu\nu} = 0$ is reduced to the equations

$$\nabla \cdot (\varphi \mathbf{w}) = 0 \tag{5.72}$$

$$\partial_t(\varphi\mathbf{w}) = -\nabla \times (\varphi\mathbf{w}\times\mathbf{v}) - \nabla\times(\mathbf{w}\times\nu) - \nabla_i G^{ij}. \tag{5.73}$$

Using the limit of the spatial part of the projection tensor

$$\perp^{ij} = -\delta^{ij} + w^i w^j,$$

the dissipative parts are expressed as

$$\nu = -\xi_T(\nabla_\perp \frac{\mu}{T} - \frac{\mu}{T}\kappa) \tag{5.74}$$

$$G^{ij} = \xi_L \frac{\mu}{T}(\nabla^{[i}w^{j]} - w^{[i}\kappa^{j]}), \tag{5.75}$$

with the curvature vector

$$\kappa \equiv (\mathbf{w} \cdot \nabla)\mathbf{w} = (\nabla \times \mathbf{w}) \times \mathbf{w}, \tag{5.76}$$

and $\nabla_\perp$ indicates the gradient with the w-component projected out. The curvature also satisfies the identity

$$\mathbf{w} \times \kappa = \nabla \times \mathbf{w} - (\mathbf{w} \cdot \nabla \times \mathbf{w})\mathbf{w},$$

which is used in $\mathbf{w} \times \nu$ and the dual of G,

$$\begin{aligned} \tilde{\mathbf{G}} &= \xi_L \frac{\mu}{T}(\mathbf{w} \cdot \nabla \times \mathbf{w})\mathbf{w} \\ \mathbf{w} \times \nu &= -\xi_T(\mathbf{w} \times \nabla \frac{\mu}{T} - \frac{\mu}{T}(\nabla \times \mathbf{w})_\perp). \end{aligned} \tag{5.77}$$

We have already discussed how ξ_T and gradients in μ/T are related to heat conduction. Now even if the thermodynamic variables are constant notice that ξ_T and ξ_L describe the production of entropy due to the curl of the field lines in the transverse and longitudinal directions respectively.

This can be intuitively understood in the wiggly string fluid. A curl that is completely perpendicular to w is found for instance in large loops lying in a plane. The loops tend to contract under tension in the direction of curvature. There is an outflow of heat due to the emission of small loops as the strings contract, so there will still be some net flow of strings ν even in the rest frame where there is no net flow of energy.

One idealized situation in which only the coefficient ξ_L applies is when each individual field line of w is an infinite straight line, and all field lines in a given plane perpendicular to some axis are pointing in the same direction. If the direction of the field lines in a plane changes as we move along the axis, the curl of w will point in the direction of w itself. If strings from one plane diffuse to an adjacent layer reconnections will lead to the production of entropy in the form of wiggles and there will be some loss of flux. This last point is perhaps easiest to understand in the limit of two layers of strings with nearly opposite directions reconnecting.

The nonrelativistic limit also makes it easy to see the connection to magnetohydrodynamics. The vector $\varphi\mathbf{w}$ is just the magnetic field $\mathbf{B}$, and from the equation of state (5.35) $\mu = \varphi$. So from (5.73) the electric field vector is equal to

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} + \mathbf{w} \times \nu + \tilde{\mathbf{G}}. \tag{5.78}$$

Bringing $\mu = \varphi$ inside the curls in (5.77),

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} + \frac{\xi_T}{T}(\nabla \times \mathbf{B})_\perp + \frac{\xi_T}{T^2}\mathbf{B} \times \nabla T + \frac{\xi_L}{T}(\nabla \times \mathbf{B})_w. \tag{5.79}$$

The first term is also in ideal magnetohydrodynamics and is due to the Lorentz boost out of the rest frame of the fluid. At low frequencies the displacement current can be neglected and Ohm's law can be written

$$\mathbf{E} = \sigma^{-1}\mathbf{J} = \sigma^{-1}\nabla \times \mathbf{B}. \tag{5.80}$$

So the coefficients ξ can be related to the electrical conductivity σ,

$$\xi = \frac{T}{\sigma}. \tag{5.81}$$

This is somewhat different from ordinary resistive magnetohydrodynamics due to the possibility of anisotropic conductivity, but also due to the presence of the temperature gradient term. In the string Eckart frame this term would vanish, but that would also restrict E to be parallel to B in the rest frame.

The origin of the difference can be seen by comparing our introduction of dissipative terms in this paper to the standard introduction of Ohm's law discussed in Sec. 5.2.2. In standard MHD the energy-momentum tensor is assumed to be separated into distinct fluid and electromagnetic parts $T_\mathrm{m} + T_\mathrm{EM}$ even out of equilibrium. The entropy is taken to only be a function of the fluid quantities, not the electromagnetic part. This makes sense in equilibrium since dependence on φ and the electromagnetic energy density cancel

$$\begin{aligned} ds &= \frac{1}{T}d\rho_\mathrm{total} - \frac{\mu}{T}d\varphi - \frac{m^a}{T}dn_a \\ &= \frac{1}{T}d\rho_m - \frac{m^a}{T}dn_a. \end{aligned} \tag{5.82}$$

But the string fluid approach taken in this paper has entropy be a function of electromagnetic sector out of equilibrium, leading to the presence of a term in the entropy current representing the diffusion of field lines (5.51). This diffusion term in the entropy current is ultimately responsible for the presence of the temperature gradient term in the electric field (5.79).

5.3.4 Stationary solutions

If a dissipative string fluid reaches a state of maximum entropy, the requirement that no further entropy be produced leads to stricter restrictions than are found in the perfect string fluid. This is a direct analogy to the stationary solutions of ordinary relativistic fluids which have among other things been taken to model rotating stars [21].

Clearly for the entropy to be conserved all of the dissipative terms leading to entropy production in (5.54) must vanish. For the components of the viscous stress $\pi^{\mu\nu}$ to vanish, the shear and expansion $\nabla_{(\mu}u_{\nu)}$ must also vanish. In particular,

$$\nabla_\mu u^\mu = 0 \tag{5.83}$$

$$w^\mu w^\nu \nabla_\mu u_\nu = 0. \tag{5.84}$$

So the conservation of entropy (5.8) and the vanishing of expansion implies s is constant in the flow direction

$$u^\mu \nabla_\mu s = 0.$$

If there are any conserved currents n_a besides the entropy clearly these must also be constant in the u direction by the same reasoning. Furthermore, using the vanishing of shear (5.84) in the expression for the divergence of F:

$$0 = w_\mu \nabla_\nu F^{\mu\nu} = -\nabla_\nu \varphi u^\nu. \tag{5.85}$$

So φ is also constant in the flow direction, and thus all thermodynamic variables must be.

To proceed we will make use of a general relation for perfect string fluids. From the contracted conservation of T,

$$w_\mu \nabla_\nu T^{\mu\nu} = 0,$$

it can be shown that the 'dual currents' $m^a w$ satisfy the relation

$$s\nabla_\mu \varphi T w^\mu + n_a \nabla_\mu \varphi m^a w^\mu = 0.$$

Incidentally, this is a fluid generalization of the dual current which appears in Carter's work on single strings [38]. For simplicity the following demonstration will consider the case where the entropy is the only current so that

$$\nabla_\mu \varphi T w^\mu = 0. \tag{5.86}$$

Beginning with the conservation of T, and making use of the relation above and the conservation of $\rho + p$ in the u direction:

$$0 = \nabla_\mu T^{\mu\nu} = (\rho + p) u^\mu \nabla_\mu u^\nu - \varphi T w^\mu \nabla_\mu \frac{\mu}{T} w^\nu - \nabla^\nu p.$$

The requirement that the diffusion vector ν vanishes implies

$$\perp^{\lambda\nu} (w^\mu \nabla_\mu w_\nu + \nabla_\nu \ln \frac{\mu}{T}) = 0, \tag{5.87}$$

so then the conservation of T can be simplified further to

$$0 = (\rho + p) u^\mu \nabla_\mu u^\nu + \varphi T \nabla^\nu \frac{\mu}{T} - \nabla^\nu p.$$

Making use of the thermodynamic identity (5.68), this implies

$$u^\mu \nabla_\mu u^\nu = \nabla^\nu \ln T, \tag{5.88}$$

which together with the vanishing of the shear of u leads to the conclusion

$$\nabla_{(\mu} \beta_{\nu)} = \nabla_{(\mu} \frac{1}{T} u_{\nu)} = 0. \tag{5.89}$$

So β is a Killing vector in equilibrium, a fact also true for ordinary fluids.

At this point, note that the orthogonal projection of $\nabla_\nu T^{\mu\nu} = 0$ leads to

$$\perp^{\lambda\mu} (u^\nu \nabla_\nu u_\mu - w^\nu \nabla_\nu w_\mu - \nabla_\nu \ln \mu) = 0. \tag{5.90}$$

The first two terms have a natural interpretation as the extrinsic curvature vector K,

$$K^\lambda \equiv h^\rho{}_\sigma \nabla_\rho h^{\sigma\lambda} = \perp^{\lambda\rho} (u^\sigma \nabla_\sigma u_\rho - w^\sigma \nabla_\sigma w_\rho).$$

So in the stationary solutions, curvature in the macroscopic worldsheets is balanced by changes in μ. This relation (5.90) was noticed already in [26] through a different line of reasoning. In our approach the similar relation (5.87) relating the curvature of the field lines to changes in μ/T is more quickly seen.

At equilibrium there is a Killing vector β in the direction of the velocity u. It will turn out there is also a preferred vector in the w direction. Using the conservation of F and (5.85)(5.86),

$$\begin{aligned} 0 = \nabla_\mu F^{\mu\nu} &= \varphi T w^\mu \nabla_\mu \frac{1}{T} u_\nu - \varphi u^\mu \nabla_\mu w^\nu \\ &= \varphi (u^\nu \nabla_\mu w_\nu - u^\nu \nabla_\nu w_\mu), \end{aligned}$$

where the Killing vector property was used in the second line. Therefore it is true that

$$u^\mu \nabla_{[\mu} \frac{\mu}{T} w_{\nu]} = 0,$$

and using the vanishing of ν and G (which depend on the other components),

$$\nabla_{[\mu} \alpha_{\nu]} = \nabla_{[\mu} \frac{\mu}{T} w_{\nu]} = 0. \tag{5.91}$$

So α and β, which were introduced earlier as derivatives of the entropy, form a natural coordinate system for the stationary fluid. The fact that their commutator vanishes can be easily proven from the conservation of F as above. Note that this is distinct from the analysis of a preferred spacelike vector appearing in [26]. There the assumption that all thermodynamic quantities are constant along the field lines w was effectively made, restricting the generality of the stationary solutions.

Finally we note that as for the case of an ordinary fluid, m^a/T for each current is constant throughout the fluid. This follows easily from the vanishing of the dissipative part of n_a in the Landau-Lifshitz frame [20].

5.3.5 Second-order theory

The theory we have been discussing is essentially an extension of the 'first-order' relativistic fluids of Eckart [18] and Landau-Lifshitz [20]. It is well known that these theories suffer certain difficulties. Hiscock and Lindblom have shown that the equilibrium states are unstable on short time scales under certain perturbations [22]. Another difficulty of first-order theories which is easily seen to be present in the current theory as well is the appearance of parabolic equations. For instance, the equation for longitudinal heat flow is given by (5.61)

$$Q_L = \kappa_L w^\mu (\nabla_\mu T - T u^\nu \nabla_\nu u_\mu).$$

For a system of straight strings at rest with no orthogonal gradients, this leads to the one-dimensional heat equation

$$\dot{T} = \frac{\kappa_L}{C} \partial_w^2 T,$$

where C is the heat capacity at constant flux

$$C \equiv \frac{\partial \rho}{\partial T}. \tag{5.92}$$

So a small perturbation in T will instantly be felt across the entire string.

The resolution to both problems for ordinary fluids [8][23] is by including second-order terms in expansion of the nonequilibrium entropy current s^μ (5.51). For instance, an additional term $-\frac{1}{2}ku^\mu Q_L^2$ for some positive coefficient k will lead to an extra term $-\kappa_L T^2 k\dot{Q}_L$ in the expression for heat conduction above (5.61). This will in turn modify the heat equation to

$$kCT^2\ddot{T} + \frac{C}{\kappa_L}\dot{T} = \partial_w^2 T,$$

which is now hyperbolic, with the speed of second sound

$$c_s^2 \equiv \frac{1}{kCT^2}. \tag{5.93}$$

As a practical matter however, there are many more possible independent second-order terms in the string fluid than in the ordinary Israel-Stewart theory. This is both due to the breaking of rotational symmetry into transverse and longitudinal directions, and also due to the presence of an extra direction in equilibrium. For instance there may be all the possible terms,

$$g_{\rho\sigma}\nu^\rho \pi_L^\sigma u^\nu, g_{\rho\sigma}\nu^\rho \pi_L^\sigma w^\nu, \tilde{\Sigma}_{\rho\sigma}\nu^\rho \pi_L^\sigma u^\nu, \ldots$$

and so on —each with an independent parameter.

Even so there are some principles which can restrict the number of independent terms. For one it should be required that the theory be invariant under changes of frame. The full entropy current s is a function of the tensors T and F, but we have expanded it about a certain arbitrary equilibrium state T_0, F_0. Expanding about a different equilibrium state should lead to the same result to the order of the highest term kept in the expansion.

Following the same approach as Israel-Stewart [8], the entropy current (5.51) is given a second-order correction S,

$$s^\mu = p\beta^\mu + \beta_\nu T^{\mu\nu} - \alpha_\nu F^{\mu\nu} - \frac{m^a}{T} n_a^\mu + S^\mu. \tag{5.94}$$

The principle of frame invariance is then that $ds^\mu = 0$ under changes of u and w.

The thermodynamic relation (5.49) may be Legendre transformed to

$$d(p\beta^\mu) = F_0^{\mu\nu} d\alpha_\nu - T_0^{\mu\nu} d\beta_\nu + n_0^\mu d(\frac{m^a}{T}). \tag{5.95}$$

So the change in s under changes of α, β is

$$ds^\mu = (T - T_0)^{\mu\nu} d\beta_\nu - (F - F_0)^{\mu\nu} d\alpha_\nu + dS^\mu,$$

and by frame invariance the change in S must be,

$$dS^\mu = \frac{\mu}{T}(F - F_0)^{\mu\nu} dw_\nu - \frac{1}{T}(T - T_0)^{\mu\nu} du_\nu.$$

Using the full decomposition of the tensors in Sec. 5.3.1, this is

$$\begin{aligned} dS^\mu =& \frac{\mu}{T}(-u^\mu \lambda^\nu + w^\mu \nu^\nu + G^{\mu\nu}) dw_\nu \\ &- \frac{1}{T}(u^\mu q_T^\nu - w^\mu \pi_L^\nu + \pi_T^{\mu\nu} + \Pi_T \perp^{\mu\nu}) du_\nu \\ &- \frac{1}{T}(\mu\lambda^\mu + Q_L u^\mu - \pi_L^\mu - \Pi_L w^\mu) w^\nu du_\nu. \end{aligned} \tag{5.96}$$

So S may include arbitrary terms which are invariant to second order under changes of frame, but it must also include terms so as to produce the change above.

Clearly it is important to know how the various quantities change with the frame. The changes du, dw to nearby equilibrium states are on the order of the field quantities themselves, as can be seen for instance in the change to the Eckart velocity in (5.64). The thermodynamic quantities ρ, φ, n_a defined through (5.38) are all invariant to first order, and thus so must be any thermodynamic quantity. Likewise $G^{\mu\nu}, \Pi_L, \Pi_T, \pi_T^{\mu\nu}$ are all invariant to first order, but the remaining dissipative fields are not:

$$\begin{aligned} d\nu^\mu &= -\varphi du_\perp^\mu \\ dq_T^\mu &= -(\rho + p) du_\perp^\mu \\ d\lambda^\mu &= -\varphi dw_\perp^\mu \\ d\pi_L^\mu &= -(\tau + p) dw_\perp^\mu \\ dQ_L &= +(\rho - \tau) w^\nu du_\nu, \end{aligned} \tag{5.97}$$

where the subscript $\perp$ indicates the change is projected orthogonal to u, w.

Even though these are not invariant, they can form the invariant combinations

$$q_T^\mu - \frac{\rho + p}{\varphi} \nu^\mu \tag{5.98}$$

$$\pi_L^\mu - \mu\lambda^\mu. \tag{5.99}$$

This is a very modest step in reducing the complexity of the second-order theory in that the five quantities in (5.97) may only appear with arbitrary parameters in the two combinations above. Note that the first combination, the invariant heat, was implicitly already used in (5.66) to relate ν to heat conduction.

The change in S (5.96) can only be produced by the noninvariant terms (5.97), and we will denote this noninvariant piece S_0. There is some ambiguity in how to split this from the invariant part of S, but we will make a choice so that S_0 vanishes in the Landau-Lifshitz frame. It can then be explicitly calculated:

$$\begin{aligned} S_0^\mu =& \frac{1}{T}(\frac{1}{2}u^\mu q_T^\nu - w^\mu \pi_L^\nu + \pi_T^{\mu\nu} + \mu w^\mu \lambda^\nu)\frac{q_{T\,\nu}}{\rho+p} + \frac{\mu}{T}(\frac{1}{2}u^\mu \lambda^\nu - w^\mu \nu^\nu)\frac{\lambda_\nu}{\varphi} \\ &- \frac{1}{T}(\frac{1}{2}Q_L u^\mu - \Pi_L w^\mu - \pi_L^\mu + \mu\lambda^\mu)\frac{Q_L}{\rho-\tau}. \end{aligned} \tag{5.100}$$

In the absence of any particle currents the longitudinal heat Q_L transforms differently from the other quantities (5.97). So its only appearance in the second-order theory is in the terms of S_0 above, with no new parameters.

Thus the coefficient k of the Q_L^2 term which leads to the speed of second sound (5.93) is

$$k = \frac{T^{-1}}{\rho-\tau} = \frac{1}{sT^2},$$

where the second equality uses the fact that there are no particle currents in the equation of state. So the speed of second sound is

$$c_s^2 = \frac{s}{C} = \frac{s}{T}\frac{\partial T}{\partial s}. \tag{5.101}$$

In a pressureless perfect string fluid this is just the expression for the ordinary longitudinal speed of sound (see for instance [24]).

In particular, recalling the wiggly string fluid equation of state (5.25)

$$\rho = \sqrt{(\mu_0\varphi)^2 + (T_H s)^2},$$

the speed of second sound is

$$c_s = \sqrt{\frac{\tau}{\rho}} = \sqrt{1 - \left(\frac{T}{T_H}\right)^2}. \tag{5.102}$$

This is again just equal to the ordinary speed of perturbations on the string, expressed in terms of the tension and mass density. And the second equality makes it clear that the speed of second sound is causal and vanishes as the temperature approaches the Hagedorn temperature. Of course for many reasons the wiggly string fluid equation of state should be understood as a toy model, but this reasonable result is at the very least a consistency check on the second-order theory.

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